Quasiconformal Mappings and Elliptic PDE's

(Spring 2013; Kari Astala)

Lectures 15.1. – 27.2; Mo 12-14 B 322,

Tu 10-12, Wed 12-14 C 123

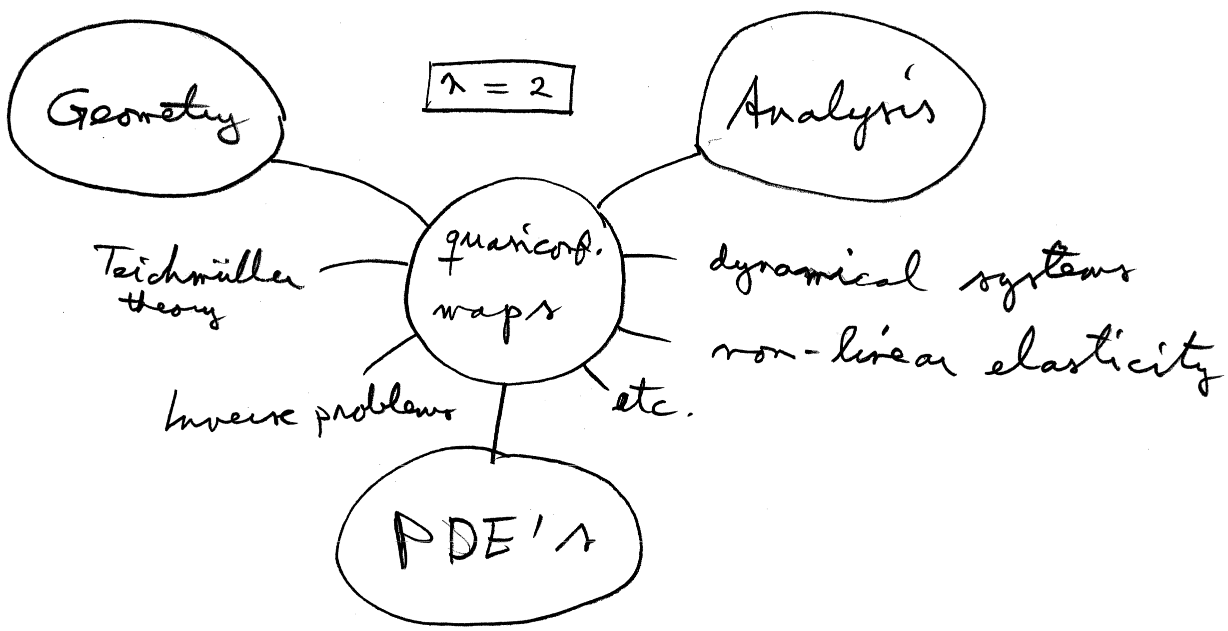
"A short introduction to quasiconformal mappings and their connections to PDE's in the plane" $(n = 2)$

Intro

Quasiconformal mappings

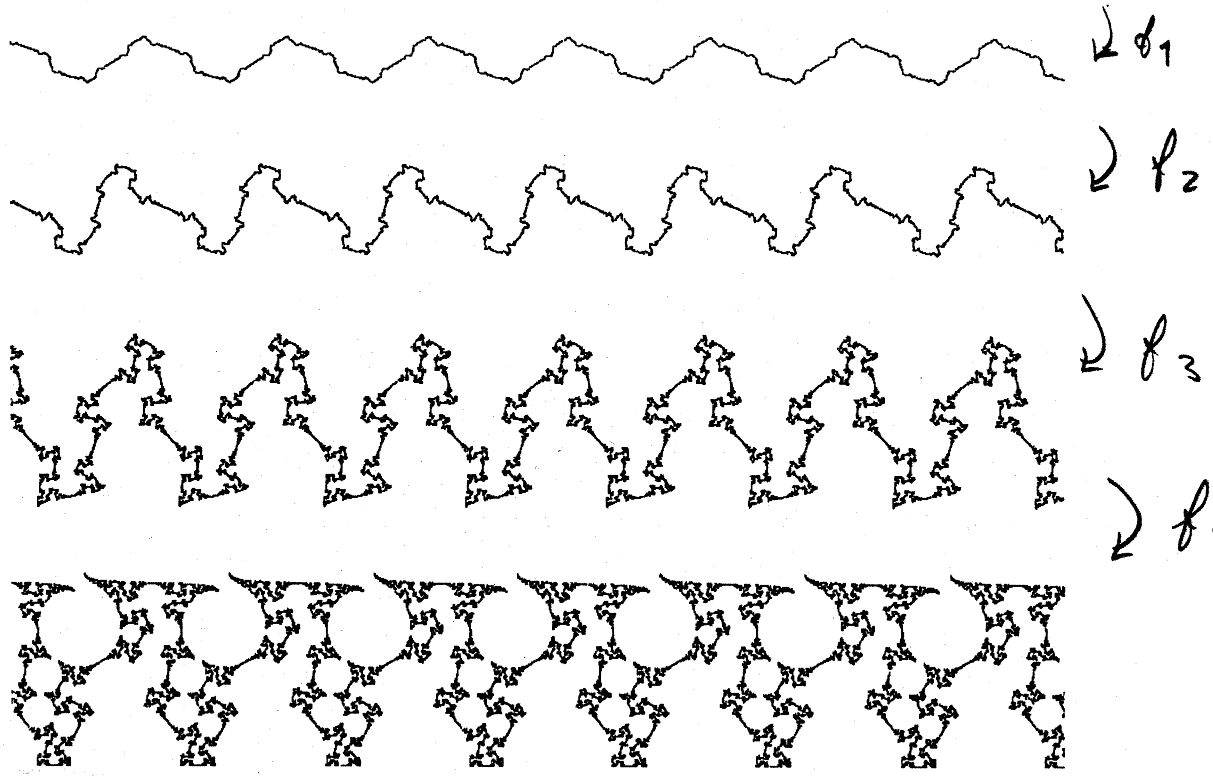
– Historically a generalization of conformal maps
⇝ name ⇝ complex analysis

– Today interacts with many branches of mathematics



We will discuss these relations in particular from following views:

<u>Geometry</u>: Quasisymmetric / Quasiconformal maps describe / govern phenomena that are "essentially" scale invariant



<u>Analysis</u>: Gives a wealth of methods; real and complex analysis, singular integrals, maximal fcns etc...

<u>Note</u>: We will use both real and complex notation!

<u>PDE's</u>: Solutions to elliptic PDE's with rough coeff.

$\equiv$ components of quasiregular / quasiconformal maps

$$\left[\ f = u + iv \iff \nabla \cdot \sigma(x) \nabla u = 0 \ ; \ \nabla \cdot \tilde{\sigma}(x) \nabla v = 0 \ \right]$$

ellipticity of σ $\iff$ QC-distortion of f.

Of these themes, the first represents a global picture of quasiconformality; the last a local or infinitesimal one.

I Quasisymmetric Maps

Before going to qc maps & PDE's want to get a picture of global geometric behaviour of qc maps; this is described by the notion of quasisymmetry.

Question: How to quantify (essential) scale invariance? (as in the picture of previous page)

Note: f a similarity $\Longleftrightarrow$ $f(z) = \alpha z + \beta$ $\alpha = re^{i\theta} \in \mathbb{C}$ (rotation + scaling) $\beta \in \mathbb{C}$ (translation)

$$\Longleftrightarrow \quad \frac{f(z)-f(w)}{f(z)-f(\xi)} = \frac{z-w}{z-\xi} \quad \forall z, w, \xi \text{ (distinct)}$$

(exercise)

1.1. Definition (Tukia-Väisälä) Let $\eta : [0,\infty) \to [0,\infty)$ be continuous & strictly increasing with $\eta(0)=0$, $\eta(t) \to \infty$ as $t \to \infty$.
A map $f : A \to \mathbb{C}$ $(A \subset \mathbb{C})$ is η-quasisymmetric if

$$(1) \qquad \frac{|f(z)-f(w)|}{|f(z)-f(\xi)|} \leq \eta\left(\frac{|z-w|}{|z-\xi|}\right) \quad \forall \text{ (distinct) } z, w, \xi \in A.$$

1.2. Remark If $\eta_0(t) \equiv t$, $t \in \mathbb{R}_+$ then

$$f \ \eta_0\text{-quasisymmetric} \Longleftrightarrow f \text{ similarity}$$

Thus "distance" of η from η_0 describes how far f is from being a similarity.

1.3. Lemma. If f is η-quasisymmetric (and $\# A \geq 3$) then

a) $f: A \to f(A)$ is a homeomorphism.

b) $f^{-1}: f(A) \to A$ is $\tilde{\eta}$-quasisymmetric, where

$$\tilde{\eta}(t) = 1 / \eta^{-1}(1/t)$$

(Note: $\tilde{\eta}$ cont., increasing, $\tilde{\eta}(0)=0$ & $\tilde{\eta}(\infty)=\infty$ whenever η satisfies these)

c) If g is η_1-quasisymmetric on $f(A)$, then $g \circ f$ is $(\eta_1 \circ \eta)$-quasisymmetric on A.

Proof: a) f is injective since for $z \neq \zeta$ (1) implies

$$\frac{|f(z) - f(w)|}{|f(z) - f(\zeta)|} \leq \eta\left(\frac{|z-w|}{|z-\zeta|}\right) < \infty \implies f(z) \neq f(\zeta)$$

$\leadsto$ injectivity. Further fixing z, ζ gives $f(w) \to f(z)$ when $w \to z$, so that continuity holds.

Continuity of f^{-1} follows from b).

b) If $z = f^{-1}a$, $w = f^{-1}b$ and $\zeta = f^{-1}c$, then

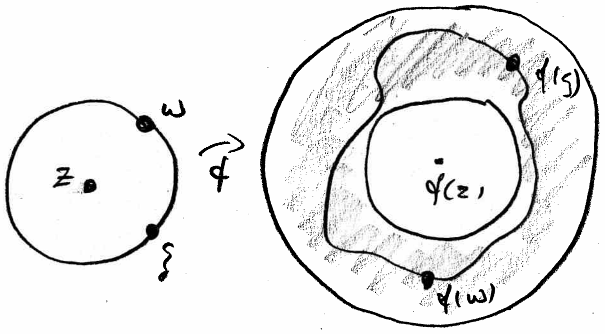
$$(1) \iff \frac{|a-b|}{|a-c|} \leq \eta\left(\frac{|z-w|}{|z-\zeta|}\right) \iff \frac{|z-\zeta|}{|z-w|} \leq \frac{1}{\eta^{-1}\left(\frac{|a-b|}{|a-c|}\right)}$$

$$\iff \frac{|f^{-1}a - f^{-1}c|}{|f^{-1}a - f^{-1}b|} \leq \tilde{\eta}\left(\frac{|a-c|}{|a-b|}\right)$$

c) Clear from definition. $\square$

Remarks. As the above proof indicates one can think η as a modulus of continuity and (1) as a similarity invariant form of this.

If $|w-z| = |\xi - z|$, then $|f(w) - f(z)| \le \eta(1)\,|f(\xi) - f(z)|$.



Thus η-quasisymmetric maps map round objects to roundish ones, and "preserve form up to η".

1.4. Examples a) If f is L-bilipschitz, i.e.

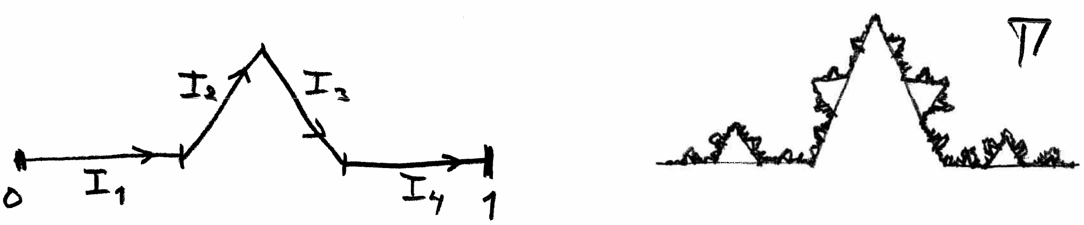
$$\frac{1}{L}|x-y| \le |f(x)-f(y)| \le L|x-y| \quad \forall\, x,y \in A, \text{ then } f \ \eta\text{-qsymm.}$$

where $\eta(t) = L^2 t$ (check!)

b) $f(z) = z|z|^{\alpha-1} = \frac{z}{|z|}|z|^{\alpha}$ is quasisymmetric

(An elementary proof possible, but an easier argument available later in discussing quasiconf. mappings)

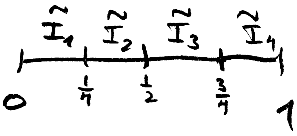
c) Consider the snowflake curve Γ:



Consider segments I_j, $j = 1, .., 4$, all of length $1/3$ as in left picture, Let ψ_j be the similarity that maps $[0,1]$ onto I_j, preserving orientation as in the picture. Then snowflake Γ is the unique compact set with

$$\Gamma = \bigcup_{j=1}^{4} \psi_j(\Gamma).$$

There is a similar structure on $[0,1]$ itself:



If $\tilde{I}_1 = [0,\frac{1}{4}]$, $\tilde{I}_2 = [\frac{1}{4},\frac{1}{2}]$, $\tilde{I}_3 = [\frac{1}{2},\frac{3}{4}]$, $\tilde{I}_4 = [\frac{3}{4},1]$,

and $\tilde{\psi}_j[0,1] = \tilde{I}_j$ [with $\tilde{\psi}_j(0) < \tilde{\psi}_j(1)$], then $[0,1]$ unique compact with

$$[0,1] = \bigcup_{j=1}^{4} \tilde{\psi}_j[0,1].$$

- There is a natural homeo $F : [0,1] \to \Gamma$ with

(2) $$F(\tilde{\psi}_j z) = \psi_j F(z), \quad z \in [0,1], \quad j = 1,..,4.$$

(For instance, if $\tilde{\psi}_{j_1} \circ \dots \circ \tilde{\psi}_{j_n}(p) = p$, i.e. p the unique fix point of $\tilde{\psi}_{j_1} \circ \dots \circ \tilde{\psi}_{j_n}$, let
$$F(p) = q \text{ where } \psi_{j_1} \circ \dots \circ \psi_{j_n}(q) = q \,)$$

Now (2) $\Rightarrow$

$$F(\tilde{\psi}_{j_1} \circ \dots \circ \tilde{\psi}_{j_n}([0,1])) = \psi_{j_1} \circ \dots \circ \psi_{j_n}(\underbrace{\Gamma}_{=F[0,1]})$$

where

- $$\mathrm{dia}(\tilde{\psi}_{j_1} \circ \dots \circ \tilde{\psi}_{j_n}[0,1]) = \frac{1}{4^n}, \quad \mathrm{dia}(\psi_{j_1} \circ \dots \circ \psi_{j_n}(\Gamma)) = \frac{1}{3^n}.$$

Since $1/3^n = (1/4^n)^\alpha$ where $\alpha = \frac{\log 3}{\log 4}$ $\left(= \frac{1}{\dim_H(\Gamma)}\right)$

it is not difficult to see that

(3) $$\frac{1}{C}|x-y|^\alpha \le |F(x) - F(y)| \le C|x-y|^\alpha \qquad \forall x,y \in [0,1]$$

(To see this scale $F(x)$, $F(y)$ and x,y until the points are fix distance apart; check the details!). It follows that

$$\frac{|F(x) - F(y)|}{|F(x) - F(z)|} \le C^2 \frac{|x-y|^\alpha}{|x-z|^\alpha} = \eta\left(\frac{|x-y|}{|x-z|}\right), \quad \eta(t) = C^2 t^\alpha$$

$\rightsquigarrow F$ η-qsymm.

Note The "Snowflake map" $F:[0,1]\to\Gamma$ of the previous example is not differentiable at any $x\in[0,1]$.

Thus quasisymmetric maps can have a large set of singularities. But how large?

In order to connect the global picture of quasisymmetry to the infinitesimal notions & PDE's, we need to answer the fundamental:

1.5. QUESTION If $f:\Omega\to\Omega'$ quasisymmetric in a domain $\Omega\subset\mathbb{R}^2$, is then f differentiable in a.e. $x\in\Omega$?

Another basic question is how to see weather a map is quasisymmetric? That is:

1.6. QUESTION How to obtain quasisymmetry from local conditions, involving the derivatives only?

These two questions are basic to (any deeper) understanding of qsymmetry. We start with the latter question and study (first) in which sense conformal mappings are quasisymmetric.

II. Background from Complex Analysis

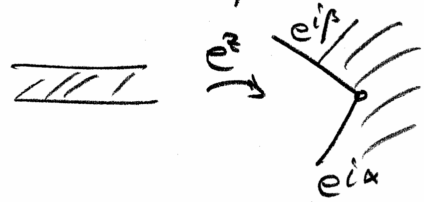
II.1. Conformal mappings

Quasiconformal mappings a generalization of conformal maps, in particular of the geometric aspects of conf. maps.

Hence we briefly recall some basics from geometric function theory.

A map $f: \Omega \to \mathbb{C}$ is conformal, if f is analytic and injective (and $\Omega \subset \mathbb{C}$ domain)

Example: For $0 < \beta - \alpha < 2\pi$, $f(z) = e^z$ is conformal in the strip $\{z : \operatorname{Im} z \in (\alpha, \beta)\}$



2.1. Remark From complex analysis we know:

If $f: \Omega \to \mathbb{C}$ analytic and $z_0 \in \Omega$, then

(i) $f'(z_0) \neq 0 \iff f$ conformal in nbhood of z_0.

(ii) If $f'(z_0) = 0$, then for some $1 < m \in \mathbb{N}$,

$$f(z) = f(z_0) + [\varphi(z)]^m \quad , \quad |z - z_0| < \varepsilon ,$$

where φ conformal in $B(z_0, \varepsilon)$ & $\varphi(z_0) = 0$.

In particular, analytic maps are open.

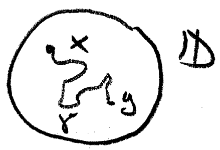
Example $f(z) = z^2$ conformal in $\mathbb{H}_+ = \{z : \operatorname{Re} z > 0\}$, but ~~not~~ in a domain Ω with $0 \in \Omega$.

<u>Remark.</u> A conformal map preserves angles infinitesimally, that is, $Df(z)h = f'(z)h$ is a similarity.

II. 2 Poincaré Metric

<u>2.2. Definition.</u> For $x, y \in \mathbb{D} = \{z : |z| < 1\}$ let

$$\rho_{\mathbb{D}}(x,y) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1-|z|^2}$$



where "inf" is taken over rectifiable curves γ connecting x to y in $\mathbb{D}$.

<u>2.3. Remark.</u> a) $\varphi : \mathbb{D} \to \mathbb{D}$ Möbius transform $\Leftrightarrow \varphi(z) = e^{i\theta}\dfrac{z-\alpha}{1-\bar{\alpha} z}$, where $\alpha \in \mathbb{D}$. $[\varphi(\alpha) = 0]$ For this map φ, derivation $\Rightarrow$

$$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \equiv \frac{1}{1-|z|^2} \quad \Rightarrow \quad \rho_{\mathbb{D}}(\varphi(z), \varphi(w)) = \rho_{\mathbb{D}}(z,w) \quad \forall z, w \in \mathbb{D}$$

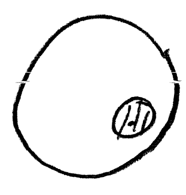
b) $\displaystyle \rho_{\mathbb{D}}(0,z) = \int_0^{|z|} \frac{2}{1-t^2}\,dt = \log \frac{1+|z|}{1-|z|}$,

and since $\varphi : \mathbb{D} \to \mathbb{D}$ is an isometry in this metric,

c) $$\rho_{\mathbb{D}}(z,w) = \log\left(\frac{1 + \frac{|z-w|}{|1-\bar{w} z|}}{1 - \frac{|z-w|}{|1-\bar{w} z|}} \right) \qquad \left(= \rho_{\mathbb{D}}(\varphi(z), \varphi(w))\right)$$

d) $$|z-w| \leq \tfrac{1}{2}(1-|z|) \Rightarrow \rho_{\mathbb{D}}(z,w) \leq \log\left(\frac{1+1/2}{1-1/2}\right) = \log 3$$

$$\rho_{\mathbb{D}}(z,w) \leq M \Rightarrow |z-w| < \delta(1-|z|), \quad \delta = \delta(M) < 1.$$



II. 3. Koebe Distortion Theorem

Koebe distortion describes general bounds valid for all conformal mappings, and from these one quickly obtains quasisymmetry in Poincare disks of bounded radius.

As a starting point we have the so called "area - theorem" which will be useful later on.

2.4. Theorem. If f is conformal in $\{|z|>1\}$ and

$f(z) = z + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$, $|z|>1$, then

$$0 \le \frac{1}{2i} \int_{|z|=R} \overline{f(z)}\, f'(z)\, dz = \pi \left(R^2 - \sum_{n=1}^{\infty} \frac{n|b_n|^2}{R^{2n}} \right), \quad R>1.$$

In particular, we have $\sum_{n=1}^{\infty} n|b_n|^2 \le 1$.

For the proof we need Green's formula in complex notation. Recall also the derivatives $\bar\partial = \partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$; $\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$

2.5. Lemma (Green's formula) If $F \in C^1(\bar\Omega)$ where $\partial\Omega$ is a C^1-Jordan curve, then

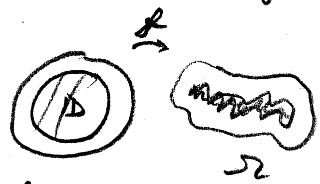
$$\frac{1}{2i}\int_{\partial\Omega} F(z)\,dz = \int_{\Omega} \bar\partial F \, dm$$

Proof: For $F = u + iv$; $LHS = \frac{1}{2i}\int_{\partial\Omega} (u+iv)(dx+idy) =$

$$\frac{1}{2i}\left(\int_{\partial\Omega} u\,dx - v\,dy\right) + \frac{1}{2}\int_{\partial\Omega} v\,dx + u\,dy \underset{\uparrow\ \text{Green}}{=} \frac{-i}{2}\int_{\Omega}(-\partial_x v - \partial_y u)\,dm \ +$$

$$+\ \frac{1}{2}\int_{\Omega}(\partial_x u - \partial_y v)\,dm \ \equiv RHS. \qquad \square$$

Proof of Theorem 2.4.: Let us prove first that the integral is positive: If $\Omega = \mathbb{C}\setminus f(|z|>R)$,



$$0 \le |\Omega| = \int_{\Omega} \partial \bar{w}\, dm \underset{\text{Le. 2.5}}{=} \frac{1}{2i}\int_{\partial\Omega} \bar{w}\, dw = \frac{1}{2i}\int_{|z|=R} \overline{f(z)}\, f'(z)\, dz$$

$\theta \mapsto f(e^{i\theta})$ parametrizes $\partial\Omega$.

To show the remaining identity plug $f(z) = z + \sum_1^\infty b_n/z^n$ into the integral $\Rightarrow$

$$\frac{1}{2i}\int_{|z|=R} \overline{f(z)}\, f'(z)\, dz = \frac{1}{2i}\int_{|z|=R}\left(\bar{z} + \sum_1^\infty \frac{\bar{b}_n}{\bar{z}^n}\right)\left(1 - \sum_{k=1}^\infty \frac{k b_k}{z^{k+1}}\right) dz$$

$$= \frac{1}{2i}\int_{|z|=R}\left(R^2 - |b_1|^2 R^{-2} - 2|b_2|^2 R^{-4} - \dots\right)\frac{dz}{z} \overset{\text{Cauchy}}{=} \pi\left(R^2 - \sum_1^\infty \frac{n|b_n|^2}{R^{2n}}\right),$$

since $\int_{|z|=R} \bar{z}^{-n} z^{-k} \frac{dz}{z} = 0$ for $n \neq k$. $\square$

With area theorem one can control the coefficients b_n. If $\varphi : \mathbb{D} \to \mathbb{C}$ is conformal in the unit disk $\mathbb{D}$, normalized by $\varphi(0) = 0$, $\varphi'(0) = 1$, one can apply Thm. 2.4 to

$$f(z) = \frac{1}{\varphi(\frac{1}{z}) - w}, \quad w \notin \varphi(\mathbb{D}),$$ and obtain bounds

for coefficients of $\varphi(z)$: E.g. if $\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $|a_2| \le 2$; i.e. $|\varphi''(0)| \le 4$. Combining φ with Möbius transforms one gets bounds for derivatives at arbitrary points $z \in \mathbb{D}$:

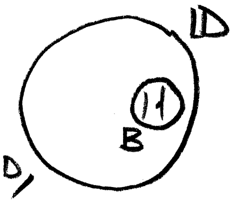
In the enclosed copies from [A-I-M] (next 4 pages) we describe how $|\varphi''(0)| \le 4$ is obtained and how the bounds give following invariant forms of Koebe distortion theorem:
[Thm. 2.10.1 in [A-I-M] is our Thm. 2.4]

<u>2.6. Theorem</u> (Thm. 2.10.8 in [A-I-M]) If f is conformal in $\mathbb{D}$ and $z, w \in \mathbb{D}$, then

$$e^{-3\rho_{\mathbb{D}}(z,w)} \le \frac{|f'(z)|}{|f'(w)|} \le e^{3\rho_{\mathbb{D}}(z,w)}$$

<u>Note:</u> By Thm. 2.6, derivatives of conformal maps are "almost" constant in disks

(2.1) $\quad B = \{z \in \mathbb{D} : |z - w| < \frac{1}{2}(1 - |w|)\}$



Integrating the bounds, see [A-I-M]/enclosed page 13.D, we have

<u>2.7. Theorem</u> If f conformal in $\mathbb{D}$ and $z_1, z_2, w \in \mathbb{D}$ with
$$\rho_{\mathbb{D}}(z_1, w) + \rho_{\mathbb{D}}(z_2, w) \le M < \infty,$$ then

$$\frac{|f(z_1) - f(w)|}{|f(z_2) - f(w)|} \le e^{4M} \frac{|z_1 - w|}{|z_2 - w|}$$

e.g. f quasisymm. in Poincare disks such as (2.1) !

13.1

2.10.2 Koebe $\frac{1}{4}$-Theorem and Distortion Theorem

The Koebe $\frac{1}{4}$-Theorem is one of the first and also one of the most powerful distortion theorems one meets in complex analysis. With the correct interpretation this result implies universal distortion estimates in hyperbolic disks, satisfied by *all* conformal mappings.

To find these we first consider the analytic function $g(z) = z+b_0+b_1z^{-1}+\cdots$, which we suppose is conformal in the exterior of the unit disk and further that $g(z) \neq w$ for $|z| > 1$. Then the branch

$$h(z) = \sqrt{g(z^2) - w} = z + \frac{1}{2}(b_0 - w)z^{-1} + \cdots$$

is well defined and conformal in the exterior disk. Furthermore, for any $r > 1$ its restriction to $\{z : |z| > r\}$ extends to a global mapping $h \in W^{1,2}_{loc}(\mathbb{C})$. Thus Theorem 2.10.1 gives

$$|w - b_0| \leqslant 2 \tag{2.67}$$

Often it is convenient to use this result in the following form.

Theorem 2.10.4. *Suppose $g : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, which is conformal in the exterior of the unit disk. If g has the development $g(z) = z+b_0+b_1z^{-1}+\cdots$ for $|z| > 1$, then*

$$g(\mathbb{D}) \subset \mathbb{D}(b_0, 2)$$

The famous Koebe $\frac{1}{4}$-theorem is a quick consequence.

Theorem 2.10.5. *Suppose that $\varphi : \mathbb{D} \to \mathbb{C}$ is conformal and normalized by $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then*

$$\mathbb{D}(0, \frac{1}{4}) \subset \varphi(\mathbb{D})$$

Proof. With our assumptions $\varphi(z) = z + a_2 z^2 + \cdots$ for $z \in \mathbb{D}$. The conjugate

$$g(z) = \frac{1}{\varphi(z^{-1})} = z - a_2 + \mathcal{O}(\frac{1}{z}), \qquad |z| > 1,$$

never vanishes, and thus (2.67) implies the classical bound of Bieberbach,

$$|a_2| \leqslant 2 \tag{2.68}$$

Also if $w \notin \varphi(\mathbb{D})$, the function

$$\varphi_1(z) = \frac{w\,\varphi(z)}{w - \varphi(z)} = z + (a_2 + \frac{1}{w})z^2 + \mathcal{O}(z^2)$$

satisfies the assumptions of the theorem, and hence we have additionally

$$|a_2 + \frac{1}{w}| \leqslant 2 \tag{2.69}$$

(13.8

Combining the bounds shows that $|w| \geqslant 1/4$ whenever $w \notin \varphi(\mathbb{D})$. □

We may view Theorem 2.10.4 as the counterpart to Koebe's result at ∞. In bounded domains the following form of Koebe's $\frac{1}{4}$-theorem applies in fact to all conformal mappings, independently of their normalization.

Theorem 2.10.6. *Suppose that f is conformal in a domain Ω with $f(\Omega) = \Omega' \subset \mathbb{C}$. Let $z_0 \in \Omega$. Then*

$$\frac{1}{4}\,|f'(z_0)|\mathrm{dist}(z_0, \partial\Omega) \leqslant \mathrm{dist}\,(f(z_0), \partial\Omega') \leqslant |f'(z_0)|\mathrm{dist}(z_0, \partial\Omega) \qquad (2.70)$$

The first inequality in (2.70) follows from Koebe's theorem, applied to

$$\varphi(z) = \frac{f(z_0 + z\,d) - f(z_0)}{d\,f'(z_0)}, \qquad d = \mathrm{dist}(z_0, \partial\Omega),$$

while the latter inequality is a consequence of the Schwarz lemma, applied to $f^{-1} : \mathbb{D}\,(f(z_0), d') \to \mathbb{D}(z_0, d)$, where $d' = \mathrm{dist}\,(f(z_0), \partial\Omega')$.

The Bieberbach bound (2.68) also provides us with uniform distortion estimates as soon as we are able to express it in an invariant form. To reveal this we introduce, for each mapping f conformal in $\mathbb{D}$, the *Koebe transform*

$$\varphi(z) = \frac{f(\frac{z+w}{1+\overline{w}z}) - f(w)}{(1-|w|^2)f'(w)}, \qquad z \in \mathbb{D} \qquad (2.71)$$

Here $w \in \mathbb{D}$ is arbitrary.

An elementary calculation gives

$$\varphi''(0) = (1-|w|^2)\frac{f''(w)}{f'(w)} - 2\overline{w} \qquad (2.72)$$

Since φ is conformal in $\mathbb{D}$ with $\varphi(0) = 0$ and $\varphi'(0) = 1$, Bieberbach's coefficient estimate yields the following theorem.

Theorem 2.10.7. *If f is conformal in the unit disk $\mathbb{D}$, then*

$$(1-|w|^2)\frac{|f''(w)|}{|f'(w)|} \leqslant 6, \qquad w \in \mathbb{D}$$

We are now in a position to prove the first of the Koebe distortion theorems. For our purposes an invariant formulation, such as the following, is the most prefered.

Theorem 2.10.8. *Suppose that f is conformal in the unit disk $\mathbb{D}$ and $z, w \in \mathbb{D}$. Then*

$$e^{-3\rho_{\mathbb{D}}(z,w)} \leqslant \frac{|f'(z)|}{|f'(w)|} \leqslant e^{3\rho_{\mathbb{D}}(z,w)}$$

13.c

Proof. Since f is conformal, the function $g(z) = \log f'(z)$ is analytic in $\mathbb{D}$. Theorem 2.10.7 tells us that $|g'(z)| \leqslant 3\, ds_{hyp}(z)$, and the claim follows via an integration,

$$\left|\log \frac{|f'(z)|}{|f'(w)|}\right| \leqslant |g(z) - g(w)| \leqslant 3\, \rho_{\mathbb{D}}(z, w) \qquad \square$$

It is remarkable that each of Theorems 2.10.5 – 2.10.8 is sharp, as the reader may verify using the Koebe function $f_0(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$. The function f_0 maps the unit disk $\mathbb{D}$ conformally onto $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

According to Theorem 2.10.8, one may consider the derivative of a conformal mapping as almost constant on hyperbolic disks! It is to be expected that then, with suitable interpretation, the mapping itself should almost be a similarity when restricted to a subdomain bounded in the hyperbolic metric.

This fact turns out to be true and is perhaps most conveniently expressed in the notation of the next theorem. Here note that a homeomorphism f in a domain Ω is a similarity if and only if

$$\frac{|f(z) - f(w)|}{|f(\zeta) - f(w)|} = \frac{|z - w|}{|\zeta - w|} \quad \text{for all } z, w, \zeta \in \Omega \tag{2.73}$$

Theorem 2.10.9. *Suppose that f is conformal in the unit disk $\mathbb{D}$. Let z_1, z_2 and $w \in \mathbb{D}$ with*

$$\rho_{\mathbb{D}}(z_1, w) + \rho_{\mathbb{D}}(z_2, w) \leqslant M < \infty$$

Then

$$\frac{|f(z_1) - f(w)|}{|f(z_2) - f(w)|} \leqslant e^{4M} \, \frac{|z_1 - w|}{|z_2 - w|} \tag{2.74}$$

Proof. We will use the Koebe transform (2.71) and evaluate $\varphi(\zeta_j)$, where $z_j = (\zeta_j + w)/(1 + \overline{w}\zeta_j)$ and $j = 1, 2$. Then $\zeta_j = (z_j - w)/(1 - \overline{w} z_j)$. Hence

$$\frac{f(z_1) - f(w)}{f(z_2) - f(w)} = \frac{\varphi(\zeta_1)}{\varphi(\zeta_2)} = \frac{z_1 - w}{z_2 - w}\, \frac{\varphi(\zeta_1)}{\zeta_1}\, \frac{\zeta_2}{\varphi(\zeta_2)}\, \frac{1 - \overline{w}\, z_2}{1 - \overline{w}\, z_1}$$

To estimate the last expression we note that

$$\begin{aligned} \log\left|\frac{1 - \overline{w}\, z_2}{1 - \overline{w}\, z_1}\right| &\leqslant \log\left(\frac{1 + \left|\frac{z_1 - w}{1 - \overline{w}\, z_1}\right|}{1 - \left|\frac{z_2 - w}{1 - \overline{w}\, z_2}\right|}\right) \\ &\leqslant \rho_{\mathbb{D}}(z_1, w) + \rho_{\mathbb{D}}(z_2, w) \end{aligned}$$

Since $\rho_{\mathbb{D}}(\zeta_j, 0) = \rho_{\mathbb{D}}(z_j, w)$, it remains to show that

$$e^{-3\, \rho_{\mathbb{D}}(\zeta, 0)} \leqslant \frac{|\varphi(\zeta)|}{|\zeta|} \leqslant e^{3\, \rho_{\mathbb{D}}(\zeta, 0)}, \qquad \zeta \in \mathbb{D} \tag{2.75}$$

C/3. D

In fact, by Theorem 2.10.8

$$|\varphi(\zeta)| = \Big|\int_0^\zeta \frac{\varphi'(z)}{\varphi'(0)} dz\Big| \leqslant \int_0^{|\zeta|} e^{3\,\rho_{\mathbb{D}}(z,0)} |dz| \leqslant |\zeta| e^{3\,\rho_{\mathbb{D}}(\zeta,0)}$$

For the former of the inequalities in (2.75), note that this is clear if $|\varphi(\zeta)| \geqslant 1/4$. Otherwise, by Koebe $\frac{1}{4}$-theorem, the interval $[\varphi(\zeta), 0] \subset \varphi(\mathbb{D})$. As $\varphi'(z)dz$ has a constant argument on $\varphi^{-1}[\varphi(\zeta), 0]$, we have

$$|\varphi(\zeta)| = \int_0^{|\zeta|} |\varphi'(z)|\,|dz| \geqslant \int_0^{|\zeta|} e^{-3\,\rho_{\mathbb{D}}(z,0)} |dz| \geqslant |\zeta| e^{-3\,\rho_{\mathbb{D}}(\zeta,0)}$$

Combining these estimates gives the inequality (2.74). □

The above theorem is an invariant version of the second Koebe distortion theorem, expressing in a compact and quantitative manner the fact that locally every conformal mapping is close to a similarity. Here, though, no claim is made on the sharpness of (2.74) in terms of the exponent $4M$. On the other hand, an important fact in Theorem 2.10.9 is the conformal invariance; via a change of variables it applies immediately to all mappings f conformal in a simply connected domain Ω.

We note also the following immediate consequence.

Corollary 2.10.10. *Conformal mappings of the plane are similarities.*

Proof. With a scaling, the estimate (2.74) holds in any disk $\mathbb{D}(0, r) = r\mathbb{D}$. If we denote $M_r = \rho_{r\mathbb{D}}(z_1, w) + \rho_{r\mathbb{D}}(z_1, w)$, then (2.74) attains the form

$$\frac{|z_1 - w|}{|z_2 - w|} e^{-4M_r} \leqslant \frac{|f(z_1) - f(w)|}{|f(z_2) - f(w)|} \leqslant \frac{|z_1 - w|}{|z_2 - w|} e^{4M_r} \tag{2.76}$$

Fixing the points z_1, z_2 and w but letting $r \to \infty$ gives $M_r = \rho_{r\mathbb{D}}(z_1, w) + \rho_{r\mathbb{D}}(z_1, w) = \rho_{\mathbb{D}}(z_1/r, w/r) + \rho_{\mathbb{D}}(z_1/r, w/r) \to 0$. Hence f satisfies (2.73). □

Bounds on the distortion of ratios such as in (2.74) quickly yield a large spectrum of various geometric properties. Indeed, the geometric study of mappings requires general notions that allow such conclusions, and for much larger classes than just (the very rigid) conformal mappings. These considerations will lead in a natural manner to the concept of quasisymmetry, which is studied and utilized in the next section. In this terminology, Theorem 2.10.9 tells us that all conformal mappings are uniformly quasisymmetric in subdomains with bounded hyperbolic diameter.

III. Analytic properties of Quasisymmetric Maps

The goal of this section is to get differentiability properties for quasisymm. maps. (at a.e. points)

III.1. Sobolev Spaces

Intuitively, the Sobolev spaces consist of those functions in $L^p(\Omega)$ which have derivatives contained in $L^p(\Omega)$.

There are several ways to make this rigorous.

3.1. Definition. If $f \in L^1_{loc}(\Omega)$, $\Omega \subset \mathbb{R}^2$ domain, we say that f has weak (or distributional) derivatives $\partial_x f \in L^1_{loc}(\Omega)$, if there is $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f(z)\, \partial_x \varphi(z)\, dm = -\int_\Omega g(z)\, \varphi(z)\, dm \quad \forall \varphi \in C_0^\infty(\Omega)$$

Write $\partial_x f := g$. Similarly for $\partial_y f$.

3.2. Definition. Let $\Omega \subset \mathbb{R}^2$ be a domain and $1 \le p \le \infty$. The Sobolev Space

$$W^{1,p}(\Omega) := \left\{ f \in L^p(\Omega) : f \text{ has weak derivatives } \partial_x f,\ \partial_y f \in L^p(\Omega) \right\}.$$

$W^{1,p}(\Omega)$ has norm $\|f\|_{1,p} := \|f\|_{L^p(\Omega)} + \|\partial_x f\|_{L^p} + \|\partial_y f\|_{L^p}$

which makes it a Banach space.

Another approach to Sobolev Spaces: Recall, that if $F:[a,b]\to\mathbb{C}$ is absolutely continuous [i.e. $\forall\varepsilon>0$ have $\delta>0$ s.t. $(x_j,y_j)\subset[a,b]$ disjoint & $\sum_{j=1}^{m}|x_j-y_j|<\delta \Rightarrow \sum_{j=1}^{m}|f(x_j)-f(y_j)|<\varepsilon$]

then if this is the case, $F'(x)$ exists for a.e. $x\in(a,b)$ and

$$(3.1)\qquad F(b)-F(a)=\int_a^b F'(t)\,dt$$

[Actually (3.1) characterizes abs. continuity]

Note: If $\Psi\in C_0^\infty(a,b) \Rightarrow \Psi\cdot F$ is abs. cont., too, and so

$$(3.2)\qquad 0=\int_a^b(\Psi F)'(t)\,dt=\int_a^b\Psi(t)F'(t)\,dt+\int_a^b\Psi'(t)F(t)\,dt.$$

3.3. Theorem. A function $f\in W^{1,p}(\Omega)$ (domain) $\iff f\in L^p(\Omega)$ and

(i) w.r.t 1-dimensional measure,

$y\mapsto f(x,y)$ abs. cont. for a.e. x,
$x\mapsto f(x,y)$ abs. cont. for a.e. y,

so that partials $\partial_y f(x,y)=\frac{d}{dy}f(x,y)$, $\partial_x f(x,y)$ exist for almost every (x,y), and

(ii) Partials from (i) satisfy $\partial_y f\in L^p(\Omega)$, $\partial_x f\in L^p(\Omega)$.

Proof: "$\Leftarrow$" is easy; use Fubini; from (3.2) above we see that derivatives from abs. continuity are weak derivatives as in Def. 3.1.

"$\Rightarrow$" More difficult; for details see book of Evans & Gariepy: "Measure theory and fine properties of fcns"

$\square$

3.4. Remark Denote
$$W^{1,p}_{loc}(\Omega) = \{ f \in W^{1,p}(U) \ \forall \text{ domain } U, \overline{U} \subset \Omega \}.$$

III. 2. Sobolev Properties of Quasisymmetric Maps.

The goal of this section is to show that quasisymmetric maps $f: \Omega \to \Omega'$ (defined in domains $\Omega, \Omega' \subset \mathbb{R}^2$ are differentiable a.e.; in fact $f \in W^{1,2}_{loc}(\Omega)$. This is non-trivial in view of Example I.1.4. c).

We start with following:

3.5. Definition. If $f \in C(\Omega)$, $\Omega \subset \mathbb{R}^2$ domain, and $\varepsilon > 0$, the maximal derivative of f is
$$L^\varepsilon_f(z) := \sup \left\{ \frac{|f(z+h) - f(z)|}{|h|} : 0 < |h| < \min\{\varepsilon, \operatorname{dist}(z, \partial\Omega)\} \right\},$$
$$z \in \Omega.$$

Note: L^ε_f is (Borel)-measurable but quite possibly $L^\varepsilon_f(z) = +\infty$ for some (or even all) $z \in \Omega$.

3.6. Lemma. If $f: \Omega \to \Omega'$ is continuous and $z_0 \in \Omega$, then for all $\varepsilon > 0$,
$$(3.3) \qquad |f(z) - f(z_0)| \le \int_\gamma L^\varepsilon_f(\zeta)\,|d\zeta|, \quad z \in \Omega,$$
for all rectifiable curves connecting z to z_0 in Ω.

Proof: Can assume γ parametrized by arc length. Then

$\int_\gamma L_f^\varepsilon(\zeta)\,|d\zeta| = \int_a^b L_f^\varepsilon(\gamma(t))\,|\gamma'(t)|\,dt$ is well defined (but possibly $= +\infty$). Fix $\varepsilon > 0$.

1°) Assume $\operatorname{diam}(\gamma) \le \varepsilon$.

Then $\forall\, \zeta \in \gamma[a,b]$

$$|f(z) - f(z_0)| \le \frac{|f(z) - f(\zeta)|}{|z-\zeta|}|z-\zeta| + \frac{|f(\zeta) - f(z_0)|}{|z_0-\zeta|}|z_0-\zeta| \le L_f^\varepsilon(\zeta)\,\ell(\gamma).$$

Integrate this along $\gamma \Rightarrow \cancel{\ell(\gamma)}\,|f(z) - f(z_0)| \le \int_\gamma L_f^\varepsilon(\zeta)\,ds \cdot \cancel{\ell(\gamma)}$

2°) If $\operatorname{diam}(\gamma) > \varepsilon$, choose $a = t_0 < t_1 < \dots < t_n = b$ such that $\operatorname{diam}(\gamma[t_j, t_{j+1}]) < \varepsilon$, $0 \le j < n$, and estimate

$$|f(z) - f(z_0)| \le \sum_{j=0}^{n-1} |f(\gamma(t_{j+1})) - f(\gamma(t_j))| \le \sum_{j=0}^{n-1} \int_{\gamma[t_j, t_{j+1}]} L_f^\varepsilon\, ds$$

$$= \int_\gamma L_f^\varepsilon\, ds. \quad \square$$

By the Lemma, L_f^ε is an <u>upper gradient</u> for f, i.e. (3.3) holds $\forall \gamma$.

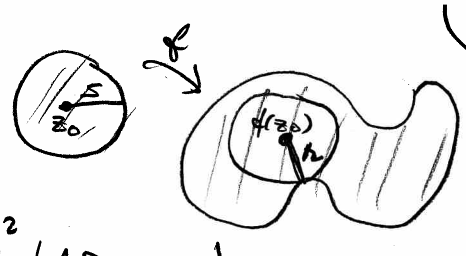
<u>3.7. Lemma</u> If $f: \Omega \to \Omega'$ is η-quasisymmetric, Ω, Ω' domains and $B(z_0, s) \subset \Omega$ disk, then

$$|f(z) - f(z_0)|^2 \le C\,|f B(z_0, s)|, \quad |z - z_0| \le s.$$

<u>Proof:</u> If $r = \min_{|\zeta - z_0| = s} |f(z_0) - f(\zeta)|$, then

$$|f(z) - f(z_0)| \le \eta(1)\, r \quad \text{whenever } |z - z_0| \le s.$$

As the disk $B(f(z_0), r) \subset f B(z_0, s)$



we have
$$|f(z) - f(z_0)|^2 \le \frac{\eta(1)^2}{\pi}\, \pi r^2 \le \frac{\eta(1)^2}{\pi}\, |f B(z_0, s)| \,. \qquad \square$$

<u>3.8. Lemma</u> If $f : \Omega \to \Omega'$ η-quasisymmetric and $B(z_0, 2r) \subset \Omega$, then $\forall\ 0 < \varepsilon < r$,

$$\left| \{ \xi \in B(z_0, r) : L_f^{\varepsilon}(\xi) > t \} \right| \le c(\eta) \frac{1}{t^2}\, |f B(z_0, r)| \,, \qquad 0 < t < \infty .$$

<u>Remark</u>: By Lemma $L_f^{\varepsilon} \in$ weak-L^2 in every disk with $2B \subset \Omega$

<u>Proof</u>: [Argument similar as showing maximal fcn is of weak type (1,1).]

Let $E_t = \{ z \in B(z_0, r) : L_f^{\varepsilon}(z) > t \}$, $0 < t < \infty$ fixed.

Then for each $x \in E_t$ can find radius $r_x \in [0, r]$ s.t.

(3.4) $\quad |f(x + r_x e^{i\theta}) - f(x)| > t\, r_x \qquad$ (for some θ).

Now $E_t \subset \bigcup_{x \in E_t} B(x, r_x)$, and using the well known $\frac{1}{5}$-covering lemma [see e.g. Holopainen: "Real Analysis I", notes] we can find a countable family of <u>disjoint</u> disks $B_j \equiv B(x_j, r_{x_j}) \subset B(z_0, 2r) \subset \Omega$ such that

$$E_t \subset \bigcup_{j=1}^{\infty} 5 B_j$$

Then

$$|E_t| \le \sum_{j=1}^{\infty} |5B_j| = 25\pi \sum_1^{\infty} r_j^2 \le$$

$$\underset{(3.4)}{\le} 25\pi \frac{1}{t^2} \sum_{j=1}^{\infty} |f(x_j + r_{x_j} e^{i\theta_j}) - f(x_j)|^2$$

$$\underset{\text{Le. 3.7}}{\le} 25\pi \frac{1}{t^2} C \sum_1^{\infty} |f B_j| \le \frac{C}{t^2} |f B(z_0, 2r)|$$

disks disjoint

$$\le C \frac{1}{t^2} |f B(z_0, r)| . \qquad \square$$

q-symmetry (& exercise)

<u>3.9. Corollary</u>. If f, ε, $B(z_0, r)$ as in Lemma 3.8, then

$$\frac{1}{|B|} \int_B (L_f^{\varepsilon})^p \, dm \le C(\eta, p) \left(\frac{|fB|}{|B|} \right)^{p/2} \qquad 1 \le p < 2, \quad B = B(z_0, r).$$

Proof: We use the well known identity (based on Fubini)

$$\int_A |g(x)|^p dx = p \int_0^{\infty} t^{p-1} |\{x \in A : |g(x)| > t\}| \, dt ,$$

with $A = B(z_0, r)$, $g = L_f^{\varepsilon}$. Thus

$$\frac{1}{|B|} \int_B (L_f^{\varepsilon})^p \, dm = \frac{p}{|B|} \int_0^{\infty} t^{p-1} |\{x \in B : L_f^{\varepsilon}(x) > t\}| \, dt$$

$$\le \frac{p}{|B|} \int_0^{t_0} t^{p-1} |B| \, dt + \frac{p}{|B|} C(\eta) \int_{t_0}^{\infty} t^{p-3} |fB| \, dt$$

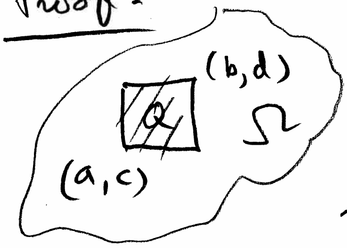
$$\le t_0^p + \frac{p\, C(\eta)}{2-p} t_0^{p-2} \frac{|fB|}{|B|} \le c \left(\frac{|fB|}{|B|} \right)^{p/2}$$

when we choose $t_0 = \sqrt{|fB|/|B|}$. $\square$

<u>3.10. Theorem</u>. If $f : \Omega \to \Omega'$ is η-quasisymmetric, $\Omega \subset \mathbb{R}^2$ (domain), then

$$f \in W^{1,1}_{loc}(\Omega).$$

<u>Proof:</u>



Let $Q := [a,b] \times [c,d] \subset \Omega$,

$\varepsilon < \operatorname{dist}(Q, \partial\Omega)$. By Corollary 3.9 & Fubini ($L_f^\varepsilon \in L^1(\Omega)$)

for a.e. $y \in [c,d]$ the integral

$x \mapsto \int_a^x L_f^\varepsilon(t,y)\,dt$ is abs. cont. on $[c,d]$

Therefore, whenever $(a_j, b_j) \subset [a,b]$ are disjoint, $j = 1, \dots, n$ we have

$$\sum_{j=1}^n |f(a_j,y) - f(b_j,y)| \underset{\text{Lemma 3.6}}{\leq} \sum_{j=1}^n \int_{a_j}^{b_j} L_f^\varepsilon(t,y)\,dt$$

$$= \int_{\bigcup_1^n (a_j,b_j)} L_f^\varepsilon(t,y)\,dt < \varepsilon, \text{ whenever } \sum_{j=1}^n |a_j - b_j| < \delta = \delta(y).$$

Thus $x \mapsto f(x,y)$ abs. cont for a.e. $y \in [c,d]$.

Similarly, $y \mapsto f(x,y)$ abs. cont. for a.e. $x \in [a,b]$.

Moreover

$$|\partial_x f(x,y)|,\ |\partial_y f(x,y)| \leq L_f^\varepsilon(x,y) \underset{\text{Corollary 3.9}}{\in} L^1(Q)$$

Thus $\partial_x f, \partial_y f \in L^1(\Omega)$ and $f \in W^{1,1}_{loc}(\Omega)$ by Thm. 3.3.

$\square$

<u>Remark.</u> $f = u + iv$ $\mathbb{C}$-valued. If necessary, can study Sobolev properties of u, v separately.

Next, want to show that any qsymm. $f \in W^{1,2}_{loc}(\Omega)$.

For this need some real analysis: Given any (see e.g. Rudin: Real and Complex Analysis) finite positive Borel measure ν on $\mathbb{R}^n$, we have the decomposition: ($A \subset \mathbb{R}^n$ Borel)

$$(3.5a) \qquad \nu(A) = \int_A g(x)\,dx + \lambda(A)$$

where

$$(3.5.b) \quad g(x) := \lim_{r \to 0} \frac{\nu(B(x,r))}{|B(x,r)|} \equiv D\nu(x) \ \exists \text{ for a.e. } x \ \& \ g \in L^1(\mathbb{R}$$

and where $\lim\limits_{r \to 0} \dfrac{\lambda(B(x,r))}{|B(x,r)|} = 0$, i.e. λ singular part of ν, $g(x)dx$ the abs. cont. part of ν.

<u>3.11. Theorem.</u> If $f : \Omega \to \Omega'$ 2-qsymm. and Ω, Ω' domain with $\Omega' = f(\Omega)$ bounded, then $f \in W^{1,2}_{loc}(\Omega)$.
In fact, we have $L_f \in L^2(\Omega)$, where

$$L_f(x) := \limsup_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} .$$

<u>Proof:</u> Note that $\varepsilon \mapsto L_f^{\varepsilon}(x)$ decreasing, and $L_f(x) = \lim\limits_{\varepsilon \to 0} L_f^{\varepsilon}$.

Lemma 3.7 $\Rightarrow$ If $B(z_0, r) \subset \Omega$,

$$(3.6) \qquad \frac{|f(z) - f(z_0)|^2}{|z - z_0|^2} \le C \frac{|f B(z_0, r)|}{|B(z_0, r)|}, \quad |z - z_0| = r$$

Considering the finite measure $\nu(A) = |f(A \cap \Omega)|$, we have a.e

$$|\partial_x f|^2, |\partial_y f|^2 \le L_f(x)^2 \underset{(3.6)}{\le} C \lim_{r \to 0} \frac{\nu(B(x,r))}{|B(x,r)|} = C D\nu(x) \underset{(3.5)}{\in} L^1(\Omega$$

$\square$

Thus $f \in W^{1,2}(\Omega)$ whenever f q-symm. & $f(\Omega)$ bdd.

In general have $f \in W^{1,2}_{loc}(\Omega)$ for any q-symm. (in a doma

We will need even a further refinement:

<u>3.12. Theorem</u> (Gehring–Lehto) If $f : \Omega \to \Omega'$ is a $W^{1,1}_{loc}$-homeomorphism, then $f(x)$ is differentiable at a.e. $x \in \Omega$.

In fact, the result remains true if $f \in W^{1,2}_{loc}(\Omega)$ is an open mapping.

(For the details of the proof see the next 4 pages, copied from [A-I-M].)

23A

First we establish a fairly well-known and general theorem of Gehring and Lehto [137] which asserts that an open mapping with finite partial derivatives at almost every point is differentiable at almost every point. For homeomorphisms the result was earlier established by Menchoff [258]. The proof must use properties of the plane, as it is false in higher dimensions, although certain analogs exist; see [180]. Second, with the Gehring-Lehto result we are able to connect the volume derivatives and the pointwise Jacobians $J(z, f)$ and thereby to obtain a first version of the change-of-variables formula.

3.3.1 The Differentiability of Open Mappings

We recall that a mapping $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is *open* if $f(U)$ is open for every open $U \subset \Omega$.

To begin with we will need the following refinements of the concept of density. Let E be a measurable subset of $\mathbb{C}$. A point $z_0 = x_0 + iy_0 \in E$ is called a point of x-density if x_0 is a point of linear density of the set $\{x \in \mathbb{R} : x + iy_0 \in E\}$. Similarly we have the notion of y-density. A point $z_0 \in E$ that is both a point of x-density and of y-density will be called a point of xy-density. Of course, as soon as we are able to establish the measurability of the set of points of xy-density, then Fubini's theorem implies that such points have full measure in E. Indeed, the set E_1 consisting of points of x-density has full measure in E for otherwise $|E \setminus E_1| > 0$ and by Fubini's theorem we could find y_0 such that the set $\{x \in \mathbb{R} : x + iy_0 \in E \setminus E_1\}$ has positive linear measure. But this would contradict the Lebesgue density theorem on the real line. Analogously, the set E_2 of points of y-density is measurable with full measure and therefore the intersection of these two sets has full measure. This is of course the set of all points of xy-density.

To show the measurability of E_1, it is enough to consider closed sets E. We denote by $E_{n,k}$ the set of points $x + iy \in E$ such that

$$\mathcal{H}^1(\{t \in [a,b] : t + iy \in E\}) \geqslant (1 - \frac{1}{n})(b - a)$$

whenever

$$a < x < b \quad \text{and} \quad 0 < b - a \leqslant \frac{1}{k}$$

Then clearly

$$E_1 = \bigcap_{n=1}^{\infty} \bigcup_{k}^{\infty} E_{n,k},$$

and it suffices to show that the sets $E_{n,k}$ are closed. Here, let $z_j = x_j + iy_j \in E_{n,k}$ with $z_j \to z_0 = x_0 + iy_0$. If $a < x_0 < b$ with $b - a < 1/k$, then $a < x_j < b$ for all j large enough, and since E is closed,

$$\begin{aligned}\mathcal{H}^1(\{t \in [a,b] : t + iy_0 \in E\}) &\geqslant \mathcal{H}^1(\cap_{\ell=1}^{\infty} \cup_{j=\ell}^{\infty} \{t \in [a,b] : t + iy_j \in E\}) \\ &\geqslant (1 - \frac{1}{n})(b - a)\end{aligned}$$

23B)

Therefore $z_0 \in E_{n,k}$ which proves that $E_{n,k}$ is closed and hence that E_1 is measurable. We argue in the same manner to see E_2 is measurable. We have thus established that the set of all points of xy-density in E is measurable with full measure.

We next have the following lemma whose proof is an elementary argument in linear density and measure theory.

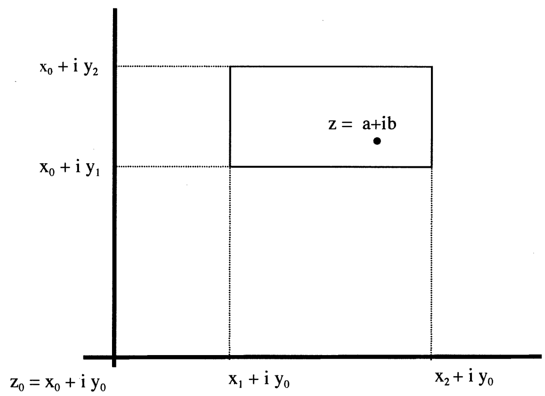
Lemma 3.3.1. *Let $\varepsilon > 0$ and $z_0 = x_0 + iy_0 \in E$ be a point of xy-density of E. Then, for all $z = a + ib$ sufficiently close to z_0, there is a rectangle*

$$R = [x_1, x_2] \times [y_1, y_2]$$

containing z and such that

$$(x_2 - x_1) < 2\,\varepsilon\,|a - x_0|, \quad (y_2 - y_1) < 2\,\varepsilon\,|b - y_0|$$

and such that the points $x_1 + iy_0$, $x_2 + iy_0$, $x_0 + iy_1$ and $x_0 + iy_2$ all lie in E.



Choosing the rectangle $R = [x_1, x_2] \times [y_1, y_2]$

Proof. We may assume that $x_0 = y_0 = 0$, and so z_0 is the point at the origin. We may also assume $a, b > 0$. Let

$$E_x = \{x \in \mathbb{R} : x + i0 \in E\}, \qquad E_y = \{y \in \mathbb{R} : 0 + iy \in E\}$$

Since $x = 0$ is a point of density of E_x, each of the intervals $(a - \varepsilon a, a)$ and $(a, a + \varepsilon a)$ contains points of E_x provided that a is sufficiently small, say $a < \delta$. We pick $x_1 \in (a - \varepsilon a, a)$ and $x_2 \in (a, a + \varepsilon a)$ such that both $x_1 + i0$ and $x_2 + i0$ lie in E. Similarly, we find $y_1 \in (b - \varepsilon b)$ and $y_2 \in (b, b + \varepsilon b)$ with both $0 + iy_1$ and $0 + iy_2 \in E$, again provided b is sufficiently small. Now we have $x_2 - x_1 < 2\varepsilon a$ and $y_2 - y_1 < 2\varepsilon b$, as desired. □

We shall now prove the following theorem.

23 5

Theorem 3.3.2. *Let $f : \Omega \to \mathbb{C}$ be a continuous open mapping. Then f is differentiable almost everywhere in Ω if and only if f has finite first partials almost everywhere.*

Proof. If f is differentiable almost everywhere, then f has finite first partials almost everywhere. It is the converse that we need to establish. Thus we assume that the partials f_x and f_y exist and are finite at almost every point of Ω. It will be enough to prove that f is in fact differentiable at almost every point of a given compact subset $X \subset \Omega$. Let t be a real number, $0 < |t| < \operatorname{dist}(X, \partial\Omega)$. We define, for those $z = x + iy$ at which both partials exist and are finite, the function

$$F_t(z) = \left|\frac{f(x+t,y) - f(x,y)}{t} - f_x(x,y)\right| + \left|\frac{f(x,y+t) - f(x,y)}{t} - f_y(x,y)\right| \tag{3.4}$$

It is easy to see that the set where F_t is defined is a Borel set [316, p. 70]. Thus F_t is a Borel function defined almost everywhere on X and it follows that the functions

$$g_n(z) = \sup_{0<|t|<1/n} F_t(z)$$

are also Borel for sufficiently large n (as it suffices to let t run through only rational values). From our assumption on the partial derivatives of f, we see that

$$g_n(z) \to 0, \quad \text{almost everywhere as } n \to \infty$$

Now by the theorems of Egoroff and Lusin, there is an increasing sequence of compact subsets $X_1 \subset X_2 \subset \cdots \subset X$ with

$$|X - \bigcup_{\nu=1}^{\infty} X_\nu| = 0$$

for which we have for each ν that the functions f_x, f_y are continuous in X_ν and

$$F_t(z) \to 0, \quad \text{uniformly on } X_\nu$$

Of course, it will now suffice to fix a ν, put $E = X_\nu$ and prove that f is differentiable at any $z_0 \in E$ that is an xy-density point of E.

Let $0 < \varepsilon < 1$. We aim to prove the estimate

$$\begin{aligned}|f(z) - f(z_0) - f_x(z_0)(x - x_0) - f_y(z_0)(y - y_0)| \\ \leqslant \varepsilon(4 + |f_x(z_0)| + |f_y(z_0)|)(|x - x_0| + |y - y_0|)\end{aligned} \tag{3.5}$$

whenever $z \in \Omega$ is sufficiently close to z_0. This is enough to ensure the differentiability of f at z_0.

Now, for all $z \in E$ sufficiently close to z_0, we have

$$|f_x(z) - f_x(z_0)| < \varepsilon, \qquad |f_y(z) - f_y(z_0)| < \varepsilon, \tag{3.6}$$

23 b)

while $|F_t(z)| < \varepsilon$ if t is small. Next

$$\begin{aligned}
&|f(z) - f(z_0) - f_x(z_0)(x - x_0) - f_y(z_0)(y - y_0)| \\
&\quad\leqslant\ |f(z) - f(x + iy_0) - f_y(x + iy_0)(y - y_0)| \\
&\qquad + |f(x + iy_0) - f(z_0) - f_x(z_0)(x - x_0)| \\
&\qquad + |f_y(x + iy_0) - f_y(z_0)||y - y_0|
\end{aligned}$$

We now assume that z is sufficiently close to z_0 so as to be able to apply (3.6). Accordingly, we arrive at the estimate

$$\begin{aligned}
&|f(z) - f(z_0) - f_x(z_0)(x - x_0) - f_y(z_0)(y - y_0)| \\
&\quad\leqslant\ F_{y-y_0}(x + iy_0)|y - y_0| + F_{x-x_0}(z_0)|x - x_0| + \varepsilon|y - y_0| \\
&\quad\leqslant\ 2\varepsilon(|x - x_0| + |y - y_0|) \qquad (3.7)
\end{aligned}$$

whenever $z = x + iy \in \Omega$ is sufficiently close to z_0 and in addition $x + iy_0 \in E$. Similarly, we have this same estimate whenever $z = x + iy \in \Omega$, $x_0 + iy \in E$ and z is sufficiently close to z_0.

Up to this point we have not used the fact that f is open, and we do so now. That f is open implies that f satisfies the maximum principle—maxima occur on the boundary. In particular, for each point $z \in \Omega$ close to z_0, let R be the rectangle given by Lemma 3.3.1. Using the maximum principle, we find that the expression

$$|f(\zeta) - f(z_0) - f_x(z_0)(u - x_0) - f_y(z_0)(v - y_0)|$$

considered as a function of $\zeta = u + iv$ takes its maximum value on the boundary of R. Hence at the maximum point $\zeta \in \partial R$,

$$\begin{aligned}
&|f(z) - f(z_0) - f_x(z_0)(x - x_0) - f_y(z_0)(x - y_0)| \\
&\quad\leqslant\ |f(\zeta) - f(z_0) - f_x(z_0)(u - x_0) - f_y(z_0)(v - y_0)| \\
&\qquad + |f_x(z_0)||u - x| + |f_y(z_0)||v - y|
\end{aligned}$$

Furthermore, for each boundary point $\zeta = u + iv \in \partial R$, either $u + iy_0 \in E$ or $x_0 + iv \in E$. In view of the estimate in (3.7),

$$|f(\zeta) - f(z_0) - f_x(z_0)(u - x_0) - f_y(z_0)(v - y_0)| \leqslant 2\varepsilon(|u - x_0| + |v - y_0|)$$

As $|u - x| \leqslant \varepsilon|x - x_0|$ and $|v - y| \leqslant \varepsilon|y - y_0|$, the above estimates prove (3.5), establishing the theorem. □

We note that in the proof we used only the maximum principle on rectangles, an apparently weaker condition than assuming f is open.

The above theorem applies, in particular, to Sobolev homeomorphisms.

Corollary 3.3.3. *Every homeomorphism $f \in W^{1,1}_{loc}(\Omega)$ is differentiable almost everywhere.*

Finally, we come to the definition of a K-quasiconformal mapping.

3.13. Definition. A homeomorphism $f: \Omega \to \Omega'$ (domain) is K-quasiconformal, if $f \in W^{1,2}_{loc}(\Omega)$ and

$$\max_\alpha |\partial_\alpha f(x)| \le K \min_\alpha |\partial_\alpha f(x)| \quad \text{a.e. } x \in \Omega.$$

Remark By Gehring-Lehto theorem a K-qc f is differentiable a.e. in Ω and therefore directional derivatives

$$\partial_\alpha f(x) = \lim_{r\to 0} \frac{f(x+re^{i\alpha}) - f(x)}{r} = \lim_{r \to 0} \frac{f(x_1 + r\cos\alpha, x_2 + r\sin\alpha) - f(x)}{r}$$

$$= Df(x)\, e^{i\alpha}$$

exist for all $\alpha \in [0, 2\pi]$ at a.e. $x \in \Omega$.

Note: For a conformal map, $Df(z)e^{i\theta} = f'(z)e^{i\theta} \Rightarrow |\partial_\alpha f(z)| = |\partial_\beta f(z)|$ $\forall \alpha, \beta$

3.14. Theorem If $f: \Omega \to \Omega'$ is η-qsymmetric, then f is K-quasiconformal with $K = \eta(1)$.

Proof:
$$\frac{|f(z+re^{i\alpha}) - f(z)|}{r} \le \eta(1) \frac{|f(z+re^{i\beta}) - f(z)|}{r}$$

which shows, in the limit $r \to 0$, that

$|\partial_\alpha f(x)| \le \eta(1)\, |\partial_\beta f(x)|$ $\forall \alpha, \beta \in [0, 2\pi]$. That $f \in W^{1,2}_{loc}$ is shown in Thm 3.11. (and that f homeo in Lemma 1.3.) $\square$

Our next major goal is the converse, that any K-quasiconformal map is quasisymmetric in disks $B(z_0, r)$ where $r < \operatorname{dist}(z_0, \partial\Omega)$; just as for conformal maps in Theorem 2.7.

Before this let us collect few very important (other) properties of quasisymmetric maps.

Lusin's property (N)

3.15. Definition. Let $f: \Omega \to \Omega'$ be a measurable mapping. We say that

- f has Lusin's property (N) if
$$|E| = 0 \Rightarrow |f(E)| = 0, \quad E \subset \Omega.$$
- f has Lusin's property (N^{-1}) if
$$|E| = 0 \Rightarrow |f^{-1}(E)| = 0, \quad E \subset \Omega'.$$

Property (N) is needed e.g. for changing variables with f, and (N^{-1}) to guarantee that $u \circ f$ is measurable for any measurable u on Ω'.

3.16. Lemma If $f:\Omega \to \Omega'$ is η-quasisymmetric, $z_0 \in \Omega$ and $|z-z_0| = r < d(z_0, \partial\Omega)$, then

$$|f(z)-f(z_0)| \le C \int_{B(z_0,r)} (L_f)^2 \, dm$$

Proof: From Theorem 3.11 we know $L_f \in L^2_{loc}(\Omega)$. Let $|z-z_0| = r$, May assume $z_0 = 0$; if $|w| = r$, quasisymmetry gives us

$$|f(z)-f(0)| \le \eta(1)\,|f(w)-f(0)| \le \eta(1)\eta(2)\,|f(w)-f(w/2)|$$

$$\underset{\text{Lemma 3.6}}{\le} \eta(1)\eta(2) \int_{w/2}^{w} L_f^{\varepsilon}$$

Integrating over circle $|w| = r$ implies

$$|f(z)-f(0)| \le \frac{C}{r} \int_{\frac{r}{2}<|w|<r} L_f^{\varepsilon} \le \frac{C}{r} \int_{B(z_0,r)} L_f^{\varepsilon}$$

Next let $\varepsilon \to 0$; by monotone convergence get

$$|f(z)-f(0)| \le \frac{C}{r} \int_{B(z_0,r)} L_f \, dm \le \frac{C}{\cancel{r}} \sqrt{\int_{B(z_0,r)} L_f^2} \; \cancel{\sqrt{|B(z_0,r)|}}$$

which gives $|f(z_0)-f(0)|^2 \le c \int_{B(z_0,r)} (L_f)^2 \, dm$ $\square$

3.17. Theorem If $f: \Omega \to \Omega'$ is η-quasisymmetric, then

$$|E| = 0 \iff |fE| = 0 \quad , \quad \forall\, E \subset \Omega \text{ measurable.}$$

Proof: "$\Rightarrow$" enough. May assume $E \subset B = B(z_0, r)$; $r < \frac{1}{4}\operatorname{dist}(z_0, \partial\Omega)$.

If $|E| = 0$, cover $E = \bigcup_{j=1}^{\infty} B_j$ by disks $B_j = B(x_j, r_j)$ with

$$\sum_{j=1}^{\infty} \operatorname{dia}(B_j)^2 < \delta$$

Using Vitali type covering lemmas (see Real Analysis I) we find a subfamily $\{\tilde{B}_k\} \subset \{B_j\}$ such that

- $\tilde{B}_k \cap \tilde{B}_{k'} = \emptyset$ for $k \neq k'$
- $E \subset \bigcup_k (5\tilde{B}_k)$

Then Quasisymmetry/Exercises $\Rightarrow$ $|f(5\tilde{B}_k)| \leq C(\eta)\, |f(\tilde{B}_k)|$.

Thus

$$|fE| \leq \sum_k |f(5\tilde{B}_k)| \leq C(\eta) \sum_k |f(\tilde{B}_k)| \leq C(\eta) \sum_k \operatorname{dia}(f\tilde{B}_k)^2$$

$$\underset{(\text{Le } 3.16)}{\leq} \; C \sum_k \int_{\tilde{B}_k} L_f^2 \underset{\text{(disks disjoint)}}{=} C(\eta) \int_{\bigcup_k \tilde{B}_k} L_f^2 .$$

Since $L_f \in L^2(B)$, $E \mapsto \int_E L_f^2$ is abs. continuous, so that

$|\bigcup_k \tilde{B}_k| \leq \sum \operatorname{dia}(\tilde{B}_k)^2 < \delta \Rightarrow \int_{\bigcup_k \tilde{B}_k} L_f^2 < \varepsilon$. Thus $E \mapsto |fE|$, too, is abs. continuous. $\square$

3.18. Corollary If $f: \Omega \to \Omega'$, $g: \Omega' \to \Omega''$ are quasisymm. (and orientation preserving), then

a) $|f(E)| = \int_E J(z,f)\, dm = \int_E \det(Df(z))$, $E \subset \Omega$.

b) $D(g \circ f)(x) = Dg(f(x))\, Df(x)$ for a.e. $x \in \Omega$.

Proof:
a) Almost everywhere $f(x+h) = f(x) + Df(x)h + h\,\varepsilon(h)$, so that $\frac{|f B(x,r)|}{|B(x,r)|} \to \det(Df(x))$ a.e., and claim follows from (3.5 a, b)), applied to measure $\nu(A) = |f(A \cap \Omega)|$.

b) Since f & g are differentiable a.e and f preserves sets of measure zero, b) follows from chain rule. ∎

3.19. Remark If $f \in W^{1,1}_{loc}(\Omega)$ is a homeomorphism, but does not necessarily satisfy Lusin's condition (N), applying (3.5 a) – (3.5 b) gives, for $\nu(A) = |f(A \cap \Omega)|$ & A Borel set,

a) $\int_E J(x,f)\, dm \leq \underbrace{|fE|}_{(E \text{ Borel})}$ $\left(J(x,f) \equiv \det(Df(x))\right)$ exists a.e.

and

b) $\int_\Omega \phi \circ f(x)\, J(x,f)\, dm \leq \int_{f\Omega} \phi(\xi)\, dm$, $\phi \in C(\Omega) \cap L^1($

Proof of b): Note first that since ϕ is assumed to be continuous,

$\phi \circ f$ is continuous. For b) note that if $E \subset \Omega$ is a Borel set and $\phi = \chi_{fE}$, a) gives

$$\int_\Omega \chi_{fE} \circ f(x) \, J(x,f) \le \int_{\Omega'} \chi_{fE}(\xi) \, dm$$

By linearity b) holds when $\phi = \Sigma_1^n a_j \chi_{fE_j}$ is a simple fcn, and since a continuous ϕ is a locally uniform limit of such functions, b) follows.

Global Quasiconformal maps are Quasisymmetric

We want to show that (according to Definition 3.13) if $f : \mathbb{C} \to \mathbb{C}$ is a $W^{1,2}_{loc}$ - homeomorphism and for a.e z,

$$(3.7) \qquad \max_\alpha |\partial_\alpha f(z)| \le K \min_\alpha |\partial_\alpha f(z)|$$

then f is quasisymmetric. Looking at the singular values of $Df(x)$ we see that (3.7) is equivalent to

$$(3.8) \qquad |Df(z)|^2 \le K \, J(z,f) \quad , \quad \text{a.e. } z \in \mathbb{C}$$

For proving the quasisymmetry we need a few auxiliary lemmas. In the first, if $u : Y \to \mathbb{R}$ is continuous with Y compact, write

$$\operatorname{osc}_u(Y) = \max_{z,w \in Y} |u(z) - u(w)|$$

3.20 Lemma (Oscillation Lemma) Suppose $u \in W^{1,2}(B(0,r))$ is continuous in $\overline{B(0,r)}$ and real valued. Then for $\rho < r$,

$$\int_\rho^r \operatorname{osc}_u (S^1(t))^2 \frac{dt}{t} \leq 2\pi \int_{\rho < |z| < r} |\nabla u|^2$$

Here $S^1(t) = \{z : |z| = t\}$.

Proof As $u \in W^{1,2}$, u is abs. continuous on almost all circles $S^1(t)$, $\rho < t < r$, [as $u \circ \exp \in W^{1,2}$ is abs. continuous on lines; see Exercises]. Thus for a.e. $t \in [\rho, r]$,

$$\operatorname{osc}_u (S^1(t)) \leq \int_0^{2\pi} |\nabla u|(te^{i\theta})\, t\, d\theta \overset{\text{Hölder}}{\leq} t \left[2\pi \int_0^{2\pi} |\nabla u|^2 (te^{i\theta})\, d\theta\right]^{1/2}$$

Squaring gives

$$\int_\rho^r \operatorname{osc}_u^2 (S^1(t)) \frac{dt}{t} \leq 2\pi \int_\rho^r t \int_0^{2\pi} |\nabla u|^2 (te^{i\theta})\, d\theta . \qquad \square$$

As a second lemma we need a preliminary version of change of variables. At the moment we need to content with the following, since we do not know yet if a qconf f has Lusin properties.

3.21. Lemma Let $f \in W^{1,2}(\mathbb{C})$ be a quasiconf. homeo. (so that (3.7) holds). Then if $v \in Lip_1(\Omega)$, we have

a) $\quad v \circ f \in W^{1,2}_{loc}(\Omega')$, $\quad \Omega' = f^{-1}(\Omega)$, and

b) $$\int_{\Omega'} |\nabla(v \circ f)|^2 \leq K \int_\Omega |\nabla v|^2$$

Proof: That $v \circ f \in W^{1,2}_{loc}(\Omega)$ follows from Thm. 3.3.

To prove b) assume first that $v \in C^\infty(\Omega)$. Then at points of differentiability of $f(z)$ we have

$$|\nabla(v \circ f)|^2(z) = |Df(z)^t|^2 \, |\nabla v|(f(z)) \leq K |\nabla v|^2 (f(z)) J(z,f)$$

and Remark 3.19 b) gives

$$\int_{\Omega'} |\nabla(v \circ f)|^2 \leq K \int_{\Omega'} |\nabla v|^2 \circ f \; J(z,f) \leq K \int_{\Omega} |\nabla v|^2 < \infty,$$

proving the claim for smooth v. In the general case, when ∇v not continuous, cannot use Remark 3.19; instead note that if $|\nabla v| \in L^2(\Omega)$, there are $v_\varepsilon \in C^\infty(\Omega)$ with $\| \nabla v - \nabla v_\varepsilon \|_{L^2} \to 0$ as $\varepsilon \to 0$. So if $\phi \in C_0^\infty(\Omega')$ then

$$\int_{\Omega'} \phi \, \nabla(v \circ f) = - \int_{\Omega'} (v \circ f) \nabla \phi = - \lim_{\varepsilon \to 0} \int_{\Omega'} (v_\varepsilon \circ f) \nabla \phi$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega} \phi \, \nabla(v_\varepsilon \circ f) \leq \lim_{\varepsilon \to 0} \|\phi\|_{L^2} \, \|\nabla(v_\varepsilon \circ f)\|_{L^2(\Omega')}$$

$v_\varepsilon \in C^\infty$

$$\leq \sqrt{K} \, \|\phi\|_{L^2(\Omega')} \, \overline{\lim_{\varepsilon \to 0}} \, \|\nabla v_\varepsilon\|_{L^2(\Omega)} = \sqrt{K} \, \|\phi\|_{L^2} \, \|v\|_{L^2}$$

Taking the supremum over $\phi \in C_0^\infty(\Omega')$ with $\|\phi\|_{L^2} = 1$ proves the claim b). $\square$

Now we are to show that for homeos $f : \mathbb{C} \to \mathbb{C}$

f quasisymmetric $\Longleftrightarrow$ f quasiconformal

Let us first fix the following negligence we made in Def. 3.13:
We will always assume that a quasiconformal mapping is orientation preserving ($\Leftrightarrow J(z,f) \geq 0$).

Thus $f : \Omega \to \Omega'$ is a K-quasiconformal map if

(3.9.a) $f \in W^{1,2}_{loc}(\Omega)$ & f homeo,

(3.9.b) $\max_\alpha |\partial_\alpha f(z)| \leq K \min_\alpha |\partial_\alpha f(z)|$, a.e. $z \in \Omega$

and (3.9.c) $J(z,f) \geq 0$.

With this, let us proceed to one of main Theorems in this course:

3.22. Theorem. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal then f is η-quasisymmetric.

Remarks: • Converse was done in Theorem 3.14

• The proof gives a highly non-optimal η. A better one is given in Corollary 4.5 below.

Proof of Theorem 3.22: We need to find $\eta = \eta_K$ with

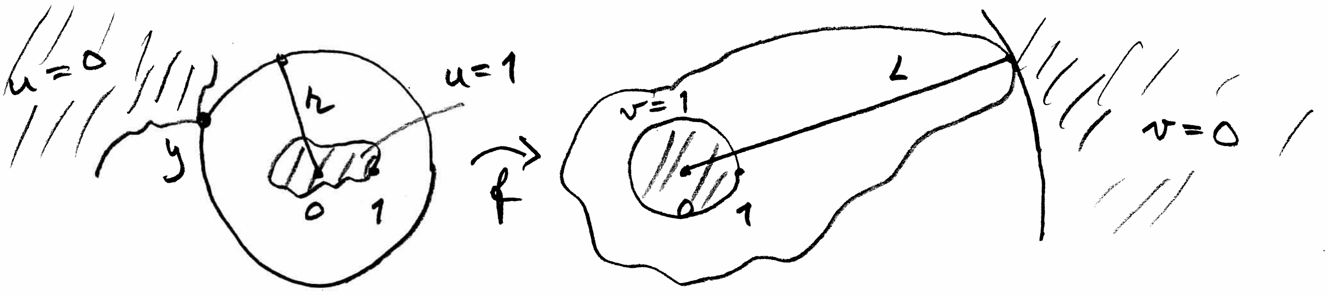
$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta\left(\frac{|x-y|}{|x-z|}\right) \quad \forall x,y,z \in \mathbb{C}.$$

Composing with similarities (which do change conditions (3.9)) may assume $x = 0 = f(x)$, $z = 1 = f(z)$. Thus need:

$$|f(y)| \leq \eta_K(|y|)$$

1°) $|y| = r \geq 1$.

Wlog, $|f(y)| = \max_{|\xi|=r} |f(\xi)| =: L > 1$.



Let $v(z) = \begin{cases} 0, & |z| \geq L \\ 1, & |z| \leq 1 \\ \dfrac{\log(|z|/L)}{\log L}, & 1 \leq |z| \leq L \end{cases}$ and $u = v \circ f$

Then

$$\int_{\mathbb{C}} |\nabla u|^2 = \int_{\mathbb{C}} |\nabla (v \circ f)|^2 \underset{\text{Le 3.2}}{\leq} K \int_{\mathbb{C}} |\nabla v|^2 = K \int_1^L \frac{2\pi}{(\log L)^2} \frac{dt}{t}$$

$\Rightarrow$

$$(3.10) \qquad \int_{\mathbb{C}} |\nabla u|^2 \leq \frac{2\pi K}{\log L}$$

To get a lower bound for $\int_{\mathbb{C}} |\nabla u|^2$, let $E = f^{-1}(\overline{B(0,1)})$ and $F = f^{-1}(\mathbb{C} \setminus B(0,L))$. Then:

i) $u|_E \equiv 1$, $u|_F \equiv 0$

ii) E, F connected; $\{0,1\} \in E$; $F \cap B(0,r) \neq \emptyset$. F unbounded

iii) $u \in W^{1,2}(\mathbb{C})$ & u continuous.

To use oscillation lemma for a lower bound need also:

3.23. Lemma. Let $|z_0| = r \geq 1$. Then $\exists\, w_0 \in \mathbb{C}$ s.t. for $\frac{r}{2} < t < \sqrt{1/2 + r^2/4}$,

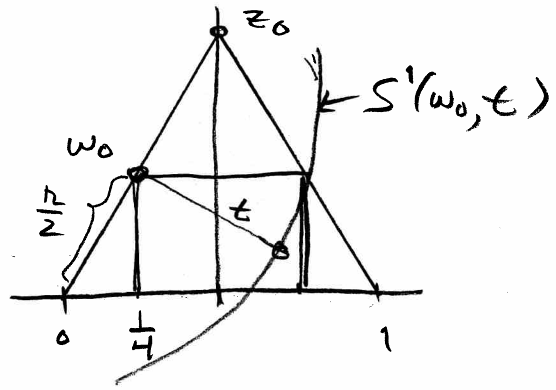
- $S^1(w_0, t) = \{\zeta : |\zeta - w_0| = t\}$ separates $0, 1$

and

- $S^1(w_0, t)$ separates z_0 from ∞.

Proof: Let first $|z_0| \leq |1 - z_0|$, take then $w_0 = \frac{z_0}{2}$.

A geometric consideration shows that extremal case is when $|z_0| = |1 - z_0|$



$$\Rightarrow \quad |w_0 - 1|^2 = \left(\tfrac{3}{4}\right)^2 + \left(\tfrac{r}{2}\right)^2 - \tfrac{1}{4}$$

Thus in general

$$|w_0 - 1| \geq \sqrt{\left(\tfrac{r}{2}\right)^2 + \tfrac{1}{2}}$$

and for $\frac{r}{2} < t < \sqrt{\left(\frac{r}{2}\right)^2 + \frac{1}{2}}$, $S^1(w_0, t)$ separates $0, 1$ and z_0, ∞.

(If $|z_0 - 1| \leq |z_0|$, claim follows by symmetry.) $\square$

Return to proof of Thm. 3.22: By Oscillation Lemma,

$$\frac{(2\pi)^2 K}{\log L} \geq 2\pi \int_{\mathbb{C}} |\nabla u|^2 \geq \int_{r/2}^{\sqrt{(r/2)^2 + 1/2}} \underbrace{\operatorname{osc}_u(S^1(w_0, t))}_{\geq 1} \frac{dt}{t}$$

$$= \frac{1}{2} \log\left(1 + \frac{1}{2r^2}\right)$$

$$\Rightarrow \quad L \leq \exp\left(\frac{8\pi^2 K}{\log\left(1 + \frac{1}{2r^2}\right)}\right) = \eta_K(r), \quad r \geq 1.$$

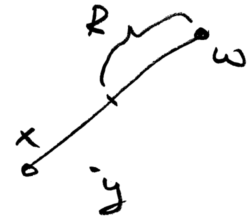
So we have shown: $\dfrac{|f(x)-f(y)|}{|f(x)-f(z)|} \le \eta_K\left(\dfrac{|x-y|}{|x-z|}\right)$ when

$\dfrac{|x-y|}{|x-z|} \ge 1$, $x, y, z \in \mathbb{C}$.

In particular f (since f homeo) satisfies the condition

$$(3.11) \qquad |z_1 - w| \le |z_2 - w| \Rightarrow |f(z_1) - f(w)| \le H\,|f(z_2) - f(w)|$$

where $H = \eta_K(1)$. This cond. is called weak quasi-symmetry. We show that for maps of $\mathbb{C}$, weak qs $\Rightarrow$ qsymmetry.



For this, if $|x-y| \le \frac{1}{2}|w-x| = R$,

Let $w_1 = \dfrac{w+x}{2}$ be midpoint of $[x,w]$.

Then weak quasisymmetry $\Rightarrow$

$$|f(x) - f(y)| \le H^2 |f(w) - f(w_1)| \le H^3 \min_{|\xi - w| = R} |f(\xi) - f(w)|$$

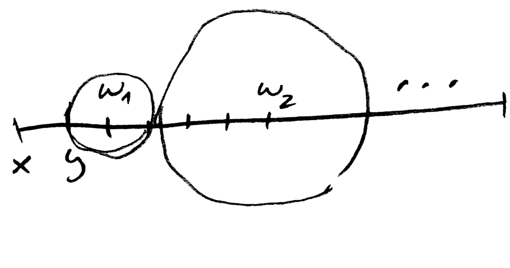
Thus $f\,B(w,R)$ contains disk of radius $\dfrac{|f(x)-f(y)|}{H^3}$

Next, if $|x - z| \equiv \rho = a^n |x-y|$, $3 < a < 9$,

can find disjoint disks

$B_j = B(w_j, R_j)$ s.t.

$R_j = \frac{1}{2}|w_j - x| > |x-y|$, $j = 1, \dots, n$



Since disks disjoint,

$$n\pi\,|f(x) - f(y)|^2 \le H^6 \sum_{j=1}^{n} |fB_j| \le H^6\,|f B(x,\rho)| \quad \le$$

$$\le \pi H^6 \max_{|\zeta - x| = \rho} |f(\zeta) - f(x)|^2 \le \pi H^8 |f(z) - f(x)|^2$$

Thus

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \le \frac{H^4}{\sqrt{n}} = \frac{H^4 \log a}{\left(\log \frac{|x-y|}{|x-z|}\right)^{1/2}} \to 0 \quad \text{as } \frac{|x-y|}{|x-z|} \to 0$$

So can take $\eta(t) = \dfrac{H^4 \log a}{(\log t)^{1/2}}$ when $0 < t < \frac{1}{3}$,

and have produced $\eta = \eta_K(t)$ for $t \ge 1$ & $0 < t < 1/3$

- On the remaining interval $\frac{1}{3} < t < 1$ can interpolate as you like, in any case have shown that any K-qc $f: \mathbb{C} \to \mathbb{C}$ is η_K-quasisymmetric.

$\square$

IV Basic Properties of Quasiconformal Maps

- Starting from Definition (3.9.a) – (3.9.c) want to prove:

4.1. Theorem If $f: \Omega \to \Omega'$ K-quasiconf. and $g: \Omega' \to \Omega''$ is K'-quasiconformal, then

(i) $f^{-1}: \Omega' \to \Omega$ is K-quasiconformal

(ii) $g \circ f: \Omega \to \Omega''$ is $K \cdot K'$-quasiconformal

(iii) $J(z, f) > 0$ for a.e. $z \in \Omega$.

(iv) $|E| = 0 \iff |f(E)| = 0 \quad \forall$ measurable $E \subset \Omega$.

Theorem 4.1 gives the basic analytic properties of quasiconformal maps. Main ingredient in the proof is of course Thm 3.22., but need also:

<u>4.2. Lemma</u> ("localization principle") If $f: \Omega \to \Omega'$ is K-quasiconformal and $\overline{B(z_0, 2r)} \subset \Omega$, then

$$f|_{B(z_0,r)} \text{ is } \eta_K\text{-quasisymm.}$$

In fact, for $z \in B(z_0,r)$, $f(z) = \varphi \circ g(z)$ where

- $g: \mathbb{C} \to \mathbb{C}$ K-quasiconf, $g B(z_0,2r) = B(z_0,2r)$
- φ conformal on $B(z_0, 2r)$

<u>Proof:</u> WLog, $B(z_0, 2r) = \mathbb{D} = \{|z| < 1\}$

Let $h: f\mathbb{D} \to \mathbb{D}$ be conformal with $h \circ f(0) = 0$ (Riemann map)

Then $h \circ f$ is K-qconformal (Check !!)

and $h \circ f: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ homeo (Caratheodory)

Extend this to all of $\mathbb{C}$ by the reflection:

Set $\Phi(z) = 1/\bar{z}$ and define

$$g(z) = \begin{cases} h \circ f(z), & |z| < 1 \\ \Phi \circ h \circ f \circ \Phi(z), & |z| > 1 \end{cases}$$

For $g \in W^{1,2}_{loc}$ see: Evans-Gariepy p.130

Then $g: \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and by $g(\mathbb{D}) = \mathbb{D}$, $g(0) = 0$, $g(\infty) = \infty$

c) $f|_{\mathbb{D}} = \varphi \circ g|_{\mathbb{D}}$, $\varphi = h^{-1} : \mathbb{D} \to f\mathbb{D}$ conformal

d) $\underbrace{\varrho_{\mathbb{D}}}_{\text{Poincaré distance}}(g(z), 0) \le M_1 = M_1(K)$, when $|z| < \frac{1}{2}$

[Proof of d): If $|z| < \frac{1}{2}$, $|\xi| = 1 \Rightarrow \frac{|g(0) - g(\xi)|}{|g(z) - g(\xi)|} \le \eta_K(2)$

$\Rightarrow \operatorname{dist}(g(z), S^1) = \delta \ge 1/\eta_K(2) > 0 \Rightarrow$

$$\varrho_{\mathbb{D}}(g(z), 0) = \log\left(\frac{1 + |g(z)|}{1 - |g(z)|}\right) \le \log \frac{2}{\delta}$$]

Now, for $z, w, x \in B(0, 1/2)$

$$\frac{|f(z) - f(w)|}{|f(x) - f(w)|} = \frac{|\varphi \circ g(z) - \varphi \circ g(w)|}{|\varphi \circ g(x) - \varphi \circ g(w)|} \overset{\text{Thm. 2.7}}{\le}$$

$$e^{M(K)} \frac{|g(z) - g(w)|}{|g(x) - g(w)|} \underset{\text{Thm 3.22.}}{\le} e^{M(K)} \eta_K\left(\frac{|z - w|}{|x - w|}\right)$$

$\square$

Proof of Thm. 4.1: By localization principle,

$f|_{B(z_0, r)}$ η_K-quasisymmetric, when $B(z_0, 2r) \subset \Omega$;

thus $|E| = 0 \Leftrightarrow |f(E)| = 0$ By Thm. 3.17.

If $E = \{z \in \Omega : J(z, f) = 0\}$, By Corollary 3.18,

$|fE| = \int_E J(z, f) = 0 \overset{\text{Thm 3.17}}{\Longrightarrow} |E| = 0$. Thus iii) & iv) hold.

i) $f|_{B(z_0,r)}$ q-symm. $\Rightarrow$ $f^{-1}|_{fB(z_0,r)}$ q-symm.

$\underset{\text{Thm 3.14}}{\Rightarrow}$ f^{-1} $\tilde{K}$-qc, for some $\tilde{K} < \infty$.

But if $z = f(w)$ and f differentiable at w, f^{-1} at z, then

$$Id = Df(w)\,Df^{-1}(z) \Rightarrow$$

$$|Df^{-1}(z)|^2 = |(Df(w))^{-1}|^2 = \frac{|Df(w)|^2}{J(w,f)^2}$$

$$\leq \frac{K}{J(w,f)} = K\,J(z,f^{-1}) \Rightarrow f^{-1} \text{ is } K\text{-qc}.$$

ii) As in i) $g \circ f$ locally q-symm. $\Rightarrow$ qconf.

For a.e. z,
$$|D(g\circ f)(z)|^2 \underset{\text{Cor. 3.18}}{\leq} |Dg(fz)|^2\,|Df(z)|^2$$

$$\leq K'\,J(fz,g)\,K\,J(z,f) = K'K\,J(z,g\circ f).$$

∎

Hölder-Continuity

As the last "basic" property let us prove that K-quasiconformal maps are $1/K$-Hölder continuous. We give a proof using the isoperimetric inequality:

4.3. Theorem (Isoperimetric Inequality). Suppose $\Omega \subset \mathbb{C}$ is a Jordan domain with $\partial\Omega$ rectifiable. Then

$$|\Omega| \leq \frac{1}{4\pi} \mathcal{H}^1(\partial\Omega)^2$$

Here $\mathcal{H}^1$ = 1-dim. Hausdorff measure ≡ length. For a proof using Fourier series see e.g. [A-I-M], p. 80.

Next, let us denote

$$\lambda(K) := \sup\{|f(e^{i\theta})| : f \text{ K-qconf on } \mathbb{C},\ f(0)=0,\ f(1)=1\}$$

$$(= \eta_K(1) \text{ for "best" } \eta_K)$$

It is known, that $1 \leq \lambda(K) \leq \frac{1}{16} e^{\pi K}$ and $\lambda(K) \to 1$ as $K \to 1$.

4.4. Theorem. If $f: \mathbb{C} \to \mathbb{C}$ is K-quasiconformal with $f(0)=0$, $f(1)=1$, then

$$|f(z)| \leq \lambda(K)^2 |z|^{1/K}, \qquad |z| < 1.$$

Moreover, $f(z) = \frac{z}{|z|} |z|^{1/K}$ is K-quasiconformal (see the next section), so that the Hölder exponent $\frac{1}{K}$ optimal.

Proof: Let $B = B(0,t)$ and write $J = J(z,f)$. Then by Isoperimetric inequality, for a.e. $t \in (0,1)$ (i.e. when $f|_{\partial B}$ is abs. cont.)

$$\int_B J \leq \frac{1}{4\pi}\Big(\int_{\partial B} |Df|\Big)^2 \underset{\text{Hölder}}{\leq} \frac{2\pi t}{4\pi} \int_{\partial B} |Df|^2 \leq \frac{Kt}{2} \int_{\partial B} J .$$

We have thus shown that $\phi(t) := \int_{B(0,t)} J(z,f)$ satisfies

$$\phi(t) \le \frac{Kt}{2}\phi'(t), \quad \text{(a.e.)} \; 0 < t \le 1.$$

$$\Rightarrow \frac{d}{dt}\left(t^{-2/K}\phi\right) = t^{-2/K-1}\left(t\phi' - \frac{2}{K}\phi\right) \ge 0, \quad \text{and as } \phi$$

is abs. cont. $\Rightarrow$

$$\phi(t) \le t^{2/K}\phi(1), \quad 0 < t \le 1.$$

Now, if $|z| = r < 1$,

$$|f(z)| \le \lambda(K) \inf_{|\zeta|=r} |f(\zeta)| \le \lambda(K)\left(\frac{1}{\pi}|fB(0,r)|\right)^{1/2}$$

$$= \lambda(K)\left(\frac{1}{\pi}\phi(r)\right)^{1/2} \le \lambda(K)\, r^{1/K}\left(\frac{1}{\pi}|fB(0,1)|\right)^{1/2}$$

$$\le \lambda(K)\, r^{1/K}\left(\max_{|\zeta|=1}|f(\zeta)|^2\right)^{1/2} \le \lambda(K)^2\, r^{1/K}. \quad \square$$

<u>4.5. Corollary</u> If $f: \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, then f is η-quasisymmetric, where

$$\eta(t) = C(K)\max\{t^K, t^{1/K}\}.$$

<u>Proof:</u> Exercises 2. $\square$

<u>4.6. Corollary</u>. If $f: \Omega \to \Omega'$ is K-quasiconformal, then f is locally $1/K$-Hölder continuous. In fact, if $2B \subset \Omega$ then

$$|f(z) - f(w)| \le C(K)\,\mathrm{diam}(fB)\frac{|z-w|^{1/K}}{\mathrm{diam}(B)^{1/K}}; \quad z, w \in B.$$

Proof: Follows from Corollary 4.5., localization principle = Lemma 4.2 and from Theorem 2.7:

$$f(z) = \underset{\text{conf. in } 2B}{\varphi} \circ \underset{K\text{-qc in } \mathbb{C}}{g(z)} \quad , z \in B .$$

$\square$

V PDE's and Quasiconformal maps in the Complex Setting.

V.1. Quasiregular Maps and Complex Notation

5.1. Definition. If $f \in W^{1,2}_{loc}(\Omega)$ satisfies

(5.1.a) $\quad \max_\alpha |\partial_\alpha f(z)| \le K \min_\alpha |\partial_\alpha f(z)| \quad$ a.e. $z \in \Omega$

and

(5.1.b) $\quad J(z,f) \ge 0 \quad$ a.e. $z \in \Omega$,

then we call f K-quasiregular.

Remarks 1°/ f K qconformal $\Leftrightarrow$ f K-qregular homeo.

2°) We will later show that each K-quasiregular map is continuous and open $\underset{\text{Gehring-Lehto}}{\Rightarrow}$ Differentiable a.e.

3°) Even if differentiability a.e. is not given by the definition, we can still define the notions in (5.1): Set $f = u + iv \Rightarrow J(z,f) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ and $\partial_\alpha f(z) = \cos(\alpha)\, \partial_x f(z) + \sin(\alpha)\, \partial_y f(z)$.

In the sequel it will convenient to use complex derivatives $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$.

E.g., by Exercise 1, if f differentiable at z, then

$$Df(z)h = \underbrace{\partial f(z)\, h}_{\substack{\mathbb{C}-\text{linear}\\ \text{in } h}} + \underbrace{\bar\partial f(z)\, h}_{\substack{\text{anti-linear}\\ \text{in } h}}, \quad ; \quad J(z,f) = |\partial f|^2 - |\bar\partial f|^2$$

Thus

$$\max_\alpha |\partial_\alpha f(z)| = |\partial f(z)| + |\bar\partial f(z)|$$

$$\min_\alpha |\partial_\alpha f(z)| = |\partial f(z)| - |\bar\partial f(z)| \qquad (\text{when } J(z,f) \geq 0)$$

so that (5.1.a) – (5.1.b) $\Longleftrightarrow$

$$|\partial f(z)| + |\bar\partial f(z)| \leq K\big(|\partial f(z)| - |\bar\partial f(z)|\big) \Longleftrightarrow |\bar\partial f| \leq \frac{K-1}{K+1} |\partial f|.$$

But this is equivalent to

$$(5.2) \qquad \begin{cases} \bar\partial f(z) = \mu(z)\, \partial f(z) & \text{for a.e. } z \in \Omega, \text{ where} \\ |\mu(z)| \leq k := \dfrac{K-1}{K+1} < 1 & \text{—''—}. \end{cases}$$

Remarks: • (5.2) is called the Beltrami equation (and has central role in the sequel)

• (5.2) is a $\mathbb{C}$-linear equation for $f = u + iv$, and it is a linear system of PDE's for u, v.

• A K-quasiregular map $f \equiv W^{1,2}_{loc}$-solution to $\bar\partial f = \mu\, \partial f$; $|\mu| \leq \frac{K-1}{K+1}$

5.2 Example. Let $f(z) = z|z|^{1/K-1} = \frac{z}{|z|}|z|^{1/K}$.

Then $\partial f(z) = \frac{1}{2}\left(\frac{1}{K}+1\right)|z|^{\frac{1}{K}-1}$ and $\bar\partial f(z) = \frac{1}{2}\left(\frac{1}{K}-1\right)\frac{z}{\bar z}|z|^{\frac{1}{K}-1}$,

(Check!)

and we have

$$\bar\partial f(z) = \frac{1-K}{1+K}\frac{z}{\bar z}\partial f(z).$$

Also $f \in W^{1,2}_{loc}(\mathbb{C})$ & homeo

Thus f is K-quasiconformal.

<u>Remark</u> For solutions to $\bar\partial f = \mu\,\partial f$ we call the coefficient $\mu(z) = \mu_f(z)$ the <u>complex</u> <u>dilatation</u> of f. Note that for quasiconformal maps (at least) $J_f(z) > 0$ a.e $\Rightarrow |\partial f|^2 - |\bar\partial f|^2 > 0$ a.e $\Rightarrow \partial f(z) \neq 0$ for a.e. z.

Thus $\mu(z) := \bar\partial f(z)/\partial f(z)$ uniquely defined almost everywhere!

V.2. On second order equations.

In this subsection Ω bounded Jordan domain.

If $\sigma(z) \in \mathbb{R}^{2\times 2}$ for a.e. $z \in \Omega$, how can one interpret the Dirichlet problem

$$\begin{cases} (5.3.a) & \nabla\cdot\sigma(z)\nabla u = 0, \quad u \in W^{1,2}(\Omega) \text{ and} \\ (5.3.b) & u|_{\partial\Omega} = F. \end{cases}$$

Note: We use both notations $\nabla \cdot E = \operatorname{div}(E) = \partial_x E^1 + \partial_y E^2$ for a vector field $E = (E^1, E^2) : \Omega \to \mathbb{R}^2$.

Let us interpret (5.3.b) first: Let

$$W_0^{1,2}(\Omega) \equiv \text{closure of } C_0^\infty(\Omega) \text{ in } \|\cdot\|_{W^{1,2}}\text{-norm}.$$

($W_0^{1,2}(\Omega) \equiv$ space of functions that vanish on $\partial\Omega$ in the Sobolev sense!!)

Assume $F \in W^{1,2}(\Omega)$ [e.g. F obtained from a function on $\partial\Omega$ by some extension procedure].

Then say for $u \in W^{1,2}(\Omega)$, that

$$u\big|_{\partial\Omega} = F \iff_{\text{def.}} u - F \in W_0^{1,2}(\Omega).$$

This takes care of (5.3.b); for (5.3.a) the eqn is interpreted in the weak sense,

$$(5.3.a) \text{ holds } \iff_{\text{def}} \int_\Omega \nabla\varphi \cdot \sigma(z) \nabla u(z)\, dm = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

Eqn (5.3) & condition $u \in W^{1,2}(\Omega)$ arises naturally in variational sense: If we want to minimize an "energy" $I(u) = \int_\Omega \langle \nabla u(z), \sigma(z) \nabla u(z) \rangle$ $[\langle a,b\rangle \equiv a\cdot b]$

with boundary values F, assuming $c_1|h|^2 \leq \langle h, \sigma(z) h\rangle \leq c_2|h|^2$ for a.e. z, we have $I(u) < \infty \iff u \in W^{1,2}(\Omega)$ and the

minimizer u_0 (i.e. $I(u_0) = \min\{ I(u) : u - F \in W_0^{1,2}(\Omega)\}$)
satisfies (5.3): For any $\varphi \in C_0^\infty(\Omega)$, when $\sigma^t \equiv \sigma$,

$$0 = \frac{d}{dt}\Big|_{t=0} \int_\Omega \langle \nabla u_0 + t\nabla\varphi, \sigma(z)(\nabla u_0 + t\nabla\varphi)\rangle = \int_\Omega \langle \nabla\varphi, \sigma\nabla u_0\rangle \, dm$$

To relate (5.3) to the Beltrami equation, need:

5.3. Lemma (Poincaré Lemma) Suppose $E \in L^p(\Omega; \mathbb{R}^2)$ is a vector field, $p \geq 1$, Ω simply connected & bounded. If $\operatorname{curl} E = 0$, i.e.

(5.4) $$\partial_x E^2 - \partial_y E^1 = 0 \qquad \text{(in the weak sense)},$$

then $E = \nabla u$ for some real valued $u \in W^{1,p}_{loc}(\Omega)$.

Proof: For $E \in C^1(\Omega; \mathbb{R}^2)$ see basic courses.
General case: Exercise set 2. $\square$

We also need the so called Hodge $*$-operator; in two dimensions it takes the simple form

$$*: \mathbb{R}^2 \xrightarrow{\text{linear}} \mathbb{R}^2, \quad * = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow ** = -I$$

i.e. $*$ = rotation by $90°$ = multiplication by i.

5.4. Remark. If $E \in L^p(\Omega; \mathbb{R}^2)$ is a vector field, then

$$\operatorname{curl} E = 0 \iff \operatorname{div}(*E) = 0 .$$

Indeed, as both operations need to be checked in the weak form, note that for $\varphi \in C_0^\infty(\Omega)$,

$$-\int_\Omega \varphi \operatorname{curl} E := \int_\Omega \partial_x \varphi \, E^2 - \partial_y \varphi \, E^1 = \int_\Omega -(\partial_x \varphi)(*E)^1 - (\partial_y \varphi)(*E)^2$$

$$=: \int_\Omega \varphi \operatorname{div}(*E) .$$

In particular, if $u \in W^{1,2}_{loc}(\Omega)$ with $\nabla \cdot \sigma(z) \nabla u(z) = 0$ (in the weak sense), then Poincaré Lemma $\Rightarrow$

$$(5.5) \qquad \nabla v = * \, \sigma(z) \nabla u(z)$$

for some $v \in W^{1,2}_{loc}(\Omega)$!!! We call v the (σ-harmonic) conjugate of u.

If $\sigma(x) \equiv Id$ then $\Delta u = 0$, u is harmonic and $v =$ harmonic conjugate of u:

$$\nabla v = * \nabla u \iff \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \iff \begin{cases} v_x = -u_y \\ v_y = u_x \end{cases} \equiv \text{Cauchy-Riemann eqn's!}$$

Thus $f = u + iv$ is analytic, i.e. $\bar\partial f = 0$.

[Little later: if $\bar\partial f = 0$ in the weak sense & $f \in W^{1,1} \Rightarrow f$ analytic]

<u>Question</u> If $v \equiv$ the σ-harmonic conjugate of u, what is the equation satisfied by $f = u + iv$?!

<u>5.5. Theorem</u>. Let $\Omega \subset \mathbb{R}^2$ be simply connected and suppose $u \in W^{1,2}_{loc}(\Omega)$ is a solution to

(5.6.a) $\quad \nabla \cdot \sigma(x) \nabla u = 0$.

If $v \in W^{1,2}_{loc}(\Omega)$ is a solution to the conjugate eqn (5.5), then $f = u + iv$ solves the (general) Beltrami equation

(5.6.b) $\quad \bar{\partial} f = \mu(z)\, \partial f + \nu(z)\, \overline{\partial f}$

where

(5.6.c) $\quad \mu(z) = \dfrac{\sigma_{22} - \sigma_{11} - i(\sigma_{12} + \sigma_{21})}{\det(I + \sigma(z))}$; $\nu(z) = \dfrac{1 - \det \sigma(z) + i(\sigma_{12} - \sigma_{21})}{\det(I + \sigma(z))}$.

Conversely, if $f \in W^{1,2}_{loc}(\Omega)$ is a mapping that satisfies (5.6.b), then $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f) \in W^{1,2}_{loc}(\Omega)$ satisfy

$$\nabla v = * \sigma(x) \nabla u \qquad [\Rightarrow \operatorname{div}(\sigma(x)\nabla u) = 0]$$

where σ and μ, ν are related by (5.7.c).

<u>Proof</u>: Elementary but tedious algebraic calculation. Some cases are discussed below, some in the Exercise set 2. $\square$

5.6. Remarks. a) For (5.6.c) to make sense we need to require, that $\det(I+\sigma(z))>0$, i.e $(1+\sigma_{11})(1+\sigma_{22})>\sigma_{12}\sigma_{21}$; but very weak ellipticity is much too weak to allow any reasonable theory. It is better to require ellipticity through (5.6.b): Require

$$(5.7) \qquad |\mu(z)|+|\nu(z)|\le k<1 \qquad \text{for a.e. } z$$

b) If $f\in W^{1,2}_{loc}(\Omega)$ satisfies $\bar\partial f=\mu\partial f+\nu\overline{\partial f}$ and (5.7) holds, then

$$|\bar\partial f(z)|\le(|\mu(z)|+|\nu(z)|)|\partial f|\le k\,|\partial f| \qquad \text{a.e. } z$$

$$\Rightarrow \quad \bar\partial f=\tilde\mu(z)\,\partial f \quad , \text{ where } \quad |\tilde\mu(z)|\le k<1 .$$

5.7. Examples a) By theorem 5.5, $\sigma(z)\equiv\sigma(z)^t \iff \nu(z)\in\mathbb{R}$ a.e. z; in this case $\nu=\dfrac{1-\det\sigma}{\det(I+\sigma)}$

b) $\sigma(x)$ is isotropic, i.e $\sigma(x)=\gamma(x)\,Id$, $\gamma(x)\in\mathbb{R} \iff \mu=0$ and $\nu(z)\in\mathbb{R}$. Then (5.6.) takes the form

$$(5.8) \qquad \bar\partial f=\frac{1-\gamma(x)}{1+\gamma(x)}\overline{\partial f} \quad , \quad \nabla\cdot\gamma\nabla u=0$$

and ellipticity (5.7) $\iff \left|\frac{1-\gamma}{1+\gamma}\right|\le k \iff \frac{1-k}{1+k}\le\gamma\le\frac{1+k}{1-k}$; that is $\gamma, 1/\gamma\in L^\infty(\Omega)$.

c) Let us check equations (5.8) are equivalent: $\nabla v=*\gamma(x)\nabla u(x)$

$$\iff \bar\partial v=i\gamma(x)\bar\partial u \iff \begin{cases}\bar\partial f=(1-\gamma)\bar\partial u\\ \overline{\partial f}=\bar\partial(u-iv)=(1+\gamma)\bar\partial u\end{cases} \iff \bar\partial f=\frac{1-\gamma}{1+\gamma}\overline{\partial f}.$$

d) Another basic case is when $\sigma(x) = \sigma(x)^t$ and $\det \sigma \equiv 1$.
This is exactly the case where $\nu(z) \equiv 0$, i.e. $f = u + iv$ satisfies the ($\mathbb{C}$-linear) equation $\bar\partial f = \mu\, \partial f$; c.f. Exercises 2.

VI. Measurable Riemann Mapping Theorem.

The goal of this is to show:

If $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \le k < 1$, then there is a homeo $f : \mathbb{C} \to \mathbb{C}$ with

- $f \in W^{1,2}_{loc}(\mathbb{C})$
- $\bar\partial f = \mu(z)\, \partial f$ a.e. $z \in \mathbb{C}$
- f is unique up to postcomposition with a similarity.

VI.1. Uniqueness

6.1. Weyl's Lemma. Suppose $f \in W^{1,1}_{loc}(\mathbb{C})$. TFAE

i) f is analytic (i.e. f has analytic representative)

ii) $\bar\partial f = 0$ in the weak sense, i.e. $\int_\Omega \bar\partial \varphi \cdot f \, dm = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$

iii) $\bar\partial f(z) = 0$ for a.e. $z \in \Omega$.

Remarks: Without condition $f \in W^{1,1}_{loc}(\Omega)$, $\bar\partial f = 0$ a.e $\not\Rightarrow$ f analytic;

simply take $f(z)=\frac{1}{z}$. BUT if we assume only $f\in L^1_{loc}(\mathbb{C})$ AND that $\bar\partial f=0$ in the weak sense, this gives f analytic ! (see proof below)

Proof of Weyl's Lemma: Clearly i) ⇒ ii), iii). Also, by Theorem 3.3, ii) ⇔ iii). Thus suffices to prove ii) ⇒ i).

For this, take $0\le\varphi\in C_0^\infty(\mathbb{C})$ with $\int_\Omega \varphi(z)\,dm=1$. Assume, say, that $\operatorname{supp}(\varphi)\subset\mathbb{D}$, and denote $\varphi_\varepsilon(z)=\frac{1}{\varepsilon^2}\varphi\left(\frac{z}{\varepsilon}\right)$.

From Real Analysis, we know: $\varphi_\varepsilon * f\in C^\infty$ and $\varphi_\varepsilon * f\to f$ (in $L^1_{loc}(\Omega)$)

More precisely, let
$$\Omega_\varepsilon=\{z\in\Omega : d(z,\partial\Omega)>\varepsilon\}.$$

Then for $z\in\Omega_\varepsilon$,
$$\bar\partial(\varphi_\varepsilon * f)(z)=\bar\partial\int_\Omega \varphi_\varepsilon(z-y)f(y)\,dm=\int_\Omega \bar\partial\varphi_\varepsilon(z-y)f(y)\overset{\text{by (ii)}}{=}0$$

Hence $\varphi_\varepsilon * f$ analytic in Ω_ε. Further, if $z\in\Omega_{2\varepsilon}$, let $B(z)=B(z,d(z,\partial\Omega)/2)$. By mean value property of analytic functions,
$$(\varphi_\varepsilon * f)(z)=⨍_{B(z)}(\varphi_\varepsilon * f)\,dm,\quad z\in\Omega_{2\varepsilon}$$

Thus
$$\left|(\varphi_\varepsilon * f)(z)-(\varphi_{\tilde\varepsilon} * f)(z)\right|\le\frac{1}{|B(z)|}\int_{B(z)}|\varphi_\varepsilon * f-\varphi_{\tilde\varepsilon} * f|\to 0\quad(\varepsilon,\tilde\varepsilon\to 0)$$

as $\varphi_\varepsilon * f\to f$ in $L^1(B(z))$.

Convergence is uniform on compact subsets of Ω,

$\Rightarrow g:=\lim_{\varepsilon\to 0}(\varphi_\varepsilon * f)(z)$ analytic in Ω, and

$$⨍_{B(z)}|g(z)-f(z)|=0\quad\forall z\ \Rightarrow\ f=g\ \text{ a.e. } z\in\Omega.\ \square$$

6.2. Remark. In particular, from Weyl's lemma and (5.2),

f 1-quasiconformal $\iff$ f conformal

Next we need to analyse the complex dilatation of a composition of quasiconformal maps ($g\circ f$). By chain rule (Exercise)

$$\partial(g\circ f) = \partial g(f)\,\partial f + \bar\partial g(f)\,\overline{\bar\partial f}\ ; \quad \bar\partial(g\circ f) = \partial g(f)\,\bar\partial f + \bar\partial g(f)\,\overline{\partial f}$$

One can write this in a matrix form:

$$\begin{pmatrix} \bar\partial(g\circ f) \\ \partial(g\circ f) \end{pmatrix} = \begin{pmatrix} \overline{\partial f} & \bar\partial f \\ \overline{\bar\partial f} & \partial f \end{pmatrix} \begin{pmatrix} \bar\partial g(f) \\ \partial g(f) \end{pmatrix}$$

and inverting the matrix gives

$$\bar\partial g(f) = \frac{1}{J(z,f)}\left(\partial f\cdot\bar\partial(g\circ f) - \bar\partial f\cdot\partial(g\circ f)\right)$$

$$\partial g(f) = \frac{1}{J(z,f)}\left(\overline{\partial f}\ \partial(g\circ f) - \overline{\bar\partial f}\cdot\bar\partial(g\circ f)\right)$$

Dividing (recall $J(z,f)>0$ for a qc map) we have

6.3. Corollary. If $f:\Omega\to\Omega'$ and $g:\Omega'\to\Omega''$ are quasiconformal, then

$$\mu_g(f) = \frac{\mu_{g\circ f} - \mu_f}{1-\mu_{g\circ f}\overline{\mu_f}}\ \frac{\partial f}{\overline{\partial f}} \qquad \text{for a.e. } z\in\Omega.$$

In other words, if we write

$$T_a(z) = \frac{z-a}{1-\bar a z}, \quad T_a: \mathbb{D}\to\mathbb{D},\ |a|<1,$$

then Corollary 6.3 gets the form:

$$\mu_g(f) = T_{\mu_f}(\mu_{g\circ f})\frac{\partial f}{\overline{\partial f}}$$

6.4. Corollary (Uniqueness part of M.R.M.T) Suppose f and h are quasiconformal in a domain $\Omega \subset \mathbb{C}$. Then following are equivalent:

(i) $\mu_f(z) = \mu_h(z)$ for a.e. $z \in \Omega$

(ii) $h = g\circ f$, where g conformal in $f(\Omega)$.

Proof: ii) ⇒ i): g smooth ⇒ chain rule applies,
$\bar\partial h = g'(f)\,\bar\partial f$, $\partial h = g'(f)\,\partial f$ ⇒ i).

i) ⇒ ii): Since f homeo, can define $g := h\circ f^{-1}$.
Theorem 4.1 ⇒ g qconf. in $f(\Omega)$, and Corollary 6.3 ⇒

$$\mu_g(fz) = \frac{\mu_h - \mu_f}{1-\overline{\mu_h}\mu_f}\frac{\partial f}{\overline{\partial f}} = 0 \text{ a.e} \Rightarrow \mu_g = 0 \text{ a.e.}$$

and by Remark 6.2 ⇒ g conformal. □

6.5. Remark If $\overbrace{f, h}^{W^{1,2}\text{ homeos}} : \mathbb{C}\to\mathbb{C}$ satisfy $\bar\partial f = \mu\,\partial f$, $\bar\partial h = \mu\,\partial h$, i.e. same Beltrami eqn, then Cor. 6.4 ⇒

$$h = \phi\circ f, \quad \text{where } \phi(z) = az+b \text{ a similarity!}$$

VI.2. Tools for Proving Existence in M.R.M.T

If $f \in C_0^1(\mathbb{C})$, then

$$(6.1) \qquad f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar\partial f(\zeta)}{(\zeta - z)}\, dm(\zeta) \quad , \quad z \in \mathbb{C}.$$

Indeed, recall Green's formula, i.e. Lemma 2.5:
$\frac{1}{2i}\int_{\partial\Omega} F(\zeta)\, d\zeta = \int_{\Omega} \bar\partial F\, dm$, $F \in C^1(\bar\Omega)$. If we let here $\Omega = B(z,R)\setminus B(z,\varepsilon)$ and $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$ and R so large that $B(z,R)$ contains $\operatorname{supp}(f)$, then: $-\frac{1}{\pi}\frac{1}{2i}\int_{\partial\Omega} F(\zeta)\, d\zeta = \frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta \to f(z)$ when $\varepsilon \to 0$ $\left[\text{as } \frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon} \frac{d\zeta}{\zeta - z} = 1\right]$.

The identity gives rise to following notion

6.6. Definition. The Cauchy transform

$$(6.2) \qquad C(h)(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(\zeta)}{\zeta - z}\, dm(\zeta) \quad , \quad \begin{array}{l} h \in C_0(\mathbb{C}), \\ z \in \mathbb{C}. \end{array}$$

The integral (6.2) converges, since $\zeta \mapsto \frac{1}{\zeta - z} \in L^q_{loc}(\mathbb{C})\ \forall q < 2$.

And by (6.1), $f = C(\bar\partial f)$ whenever $f \in C^1(\mathbb{C})$ has compact support.

6.6. Remarks. 1°) $\mathcal{C} : C_0^\infty(\mathbb{C}) \to C^\infty(\mathbb{C})$

Indeed, since $\mathcal{C}$ is a convolution operator, it commutes with all derivatives, $\partial_{x_j} \mathcal{C} = \mathcal{C}\, \partial_{x_j}$ [check this!].
But $\mathcal{C}(h)$ need not have compact support even if h does. (e.g. if $\int_{\mathbb{C}} h\, dm \neq 0$). On the other hand, $\mathcal{C}(h)(z)$ is analytic outside the support of h.

2°) For $g \in C_0^1(\mathbb{C})$,

$$(6.3) \qquad \bar\partial\left[\mathcal{C}(g)\right] = \mathcal{C}(\bar\partial g) \overset{(6.1)}{=} g .$$

In other words, $\boxed{\bar\partial \circ \mathcal{C} = \mathcal{C} \circ \bar\partial = Id}$ (on $C_0^1(\mathbb{C})$ at least)

6.7. Definition. For $g \in C_0^\infty(\mathbb{C})$ the Beurling transform is defined by

$$(6.4) \qquad S(g) := \partial\left[\mathcal{C}(g)\right] .$$

Remark For $g \in C_0^\infty(\mathbb{C})$, $z \in \mathbb{C}$, we have

$$(Sg)(z) = \partial(\mathcal{C}g)(z) = \mathcal{C}(\partial g)(z) = \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{\mathbb{C} \setminus B(z,\varepsilon)} \frac{\partial g(\zeta)}{\zeta - z}\, dm$$

$$\underset{\text{int. by parts}}{=} \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{\mathbb{C} \setminus B(z,\varepsilon)} \frac{g(\zeta)}{(\zeta - z)^2}\, dm + \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{|\zeta - z| = \varepsilon} \frac{g(\zeta)}{\zeta - z}\, d\bar\zeta ,$$ where

we have used the Green's formula related to ∂-derivative,

$$\int_{\Omega} \partial F(\zeta)\, dm = -\frac{1}{2i} \int_{\partial\Omega} F(\zeta)\, d\bar{\zeta}$$

(follows from Lemma 2.5 by taking the complex conjugate)

Above

$$\int_{|\zeta - z| = \varepsilon} \frac{g(\zeta)}{\zeta - z}\, d\bar{\zeta} = \underbrace{\int_{|\zeta - z| = \varepsilon} \frac{g(\zeta) - g(z)}{\zeta - z}\, d\bar{\zeta}}_{\leq \|g\|_{Lip_1} |2\pi\varepsilon| \to 0} + \underbrace{g(z) \int_{|\zeta - z| = \varepsilon} \frac{d\bar{\zeta}}{\zeta - z}}_{g(z) \int_0^{2\pi} \frac{-i e^{-i\theta}\, d\theta}{\varepsilon e^{i\theta}} = 0\,!} \xrightarrow{\text{as } \varepsilon \to 0} 0$$

Thus for $g \in C_0^\infty(\mathbb{C})$

$$(6.5) \quad (Sg)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\zeta - z| > \varepsilon} \frac{g(\zeta)}{(\zeta - z)^2}\, dm =: -\frac{1}{\pi}\, \text{p.v.} \int_{\mathbb{C}} \frac{g(\zeta)}{(\zeta - z)^2}\, dm$$

where p.v. stands for "principal value", defined by (6.5). Indeed the kernel $\frac{1}{(\zeta - z)^2}$ is too singular (i.e. $\notin L^1_{loc}$) for the integral $\int_{\mathbb{C}} g(\zeta)/(\zeta - z)^2$ to exist in the usual sense. But the principal value converges at every z when $g \in C_0^\infty(\mathbb{C})$. This can also be seen directly: If $g \in C_0^\infty(\mathbb{C})$ & $\operatorname{supp}(g) \subset B(z, R)$,

$$\int_{\varepsilon < |\zeta - z|} \frac{g(\zeta)}{(\zeta - z)^2} = \int_{\varepsilon < |\zeta - z| < R} \frac{g(\zeta) - g(z)}{(\zeta - z)^2}$$

which converges when $\varepsilon \to 0$, since $\frac{g(\zeta) - g(z)}{(\zeta - z)^2} \in L^1_{loc}(\mathbb{C})$

why !?