Department of Mathematics and Statistics
Quasiconformal mappings and elliptic PDE's
Exercise set 3.
19.2.2013

1. Calculate the Cauchy transform $\mathcal{C}h(z)$ and the Beurling transform $\mathcal{S}h(z)$ when

i) $h(z) = \chi_B(z_0, r)$, $z_0 \in \mathbb{C}$ and $r > 0$,

ii) $h(z) = (z/\overline{z})\chi_{\mathbb{D}}(z)$.

[Hint: Recall the example on p. 60 of the lecture notes.]

2. Suppose $2 < p < \infty$ and that $h \in L^p(\mathbb{C})$. Define the Cauchy integral operator $\mathcal{C}h$ by

$$(\mathcal{C}h)(z) = -\frac{1}{\pi}\int_{\mathbb{C}}\left[\frac{1}{\zeta - z} - \frac{\chi_{\mathbb{C}\setminus\mathbb{D}}(\zeta)}{\zeta}\right]h(\zeta)dm(\zeta)$$

Show that $\mathcal{C}h$ is well defined, and that $|\mathcal{C}h(z) - \mathcal{C}h(w)| \le C|z - w|^{1-2/p}$ for all $z, w \in \mathbb{C}$.

3. Let $\|S\|_p = \sup\{\|S(g)\|_{L^p} : \|g\|_{L^p} = 1\}$ be the operator-norm of the Beurling transform on $L^p(\mathbb{C})$. Show for all $2 \le p < \infty$ that $\|S\|_p \ge p - 1$.

[Hint: For each $\alpha > -1/p$ calculate the L^p-norms of ∂u and $\overline{\partial} u$, where $u(z) = z(|z|^{2\alpha} - 1)$ for $\varepsilon < |z| < 1$ with $u(z) = 0$ for $|z| \ge 1$ and $u(z) = z(\varepsilon^{2\alpha} - 1)$ for $|z| \le \varepsilon$. (Why case $\alpha < -1/p$ is not useful ?)]

4. Consider the mapping $f(z) = z|z|^{2i}$, for $z \in \mathbb{C}$, with $f(1) = 1$. Sketch the arc $f([-1, 1])$. Show that the inverse $f^{-1}(z) = \overline{f(\overline{z})}$ and that f is a bilipschitz map of $\mathbb{C}$.

Hence f is also a quasiconformal map of $\mathbb{C}$; determine the complex dilatation $\mu_f(z)$ and the smallest number $1 \le K$ for which f is K-quasiconformal.

[Hint: Recall that complex exponents are defined by $x^\alpha = \exp(\alpha \log(x))$ for $x \in \mathbb{R}_+$ and $\alpha \in \mathbb{C}$.]

5. Let μ and ν be functions in $L^\infty(\mathbb{C})$ such that $|\mu(z)| + |\nu(z)| \leq k < 1$ for a.e. $z \in \mathbb{C}$, for a constant $0 \leq k < 1$. Assume also that for $|z| > 1$ they vanish, $\mu(z) = \nu(z) = 0$ when $|z| > 1$.

Show that the equation

$$\overline{\partial} f(z) = \mu(z)\partial f(z) + \nu(z)\overline{\partial f(z)}, \qquad a.e.\, z \in \mathbb{C},$$

has a unique solution $f \in W^{1,2}_{loc}(\mathbb{C})$ such that $f(z) = z + \mathcal{O}(1/z)$ as $z \to \infty$.

Show also using results from the lecture notes, that the solution f is a homeomorphism of $\mathbb{C}$.