Department of Mathematics and Statistics
Quasiconformal mappings and elliptic PDE's
Exercise set 2.
12.2.2013

1. Consider the radial mapping $f(z) = z\phi(|z|^2)$, where $\phi : (0,\infty) \to (0,\infty)$ is a continuously differentiable function such that $t\phi(t^2) \to 0$ when $t \to 0$. Calculate the derivatives $\overline{\partial} f(z)$ and $\partial f(z)$, $z \neq 0$ and determine the Jacobian determinant $J_f(z)$, $z \neq 0$.

Show that f is a homeomorphism (onto its image) if $2s\phi'(s) + \phi(s) > 0$ for all $0 < s < \infty$, and that in this case $J_f(z) > 0$, $z \neq 0$.
[Hint: Recall that $|z|^2 = z\overline{z}$ and use Problem 1/exercise set 1.]

2. If $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal mapping, show that f is η-quasisymmetric where

$$\eta(t) = C \max\{t^K, t^{1/K}\}, \qquad t \geq 0,$$

for some constant $C = C(K)$ depending only on K.

[Hint: Recall our Hölder estimates. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal fixing 0 and 1, so is $f^{-1}(z)$ and $1/f(1/z)$.]

3. Prove the Poincare lemma, Lemma 5.3 in the lecture notes:

If Ω is simply connected and if $E \in L^p(\Omega, \mathbb{R}^2)$ is a L^p-vector field with $\text{curl}E = 0$ (in the weak/distributional sense), then $E = \nabla u$ for some real valued $u \in W^{1,p}_{loc}(\Omega)$.

[Hint: You may assume the result known for $E \in C^\infty(\Omega, R^2) \cap L^p(\Omega, \mathbb{R}^2)$.]

4. Suppose $\sigma : \Omega \to \mathbb{R}^{2\times 2}$ is strongly elliptic, in the sense that for some $K \geq 1$,

$$\frac{1}{K}|h|^2 \leq \langle h, \sigma(x)h\rangle \leq K|h|^2, \qquad h \in \mathbb{R}^2, \quad \text{a.e. } x \in \Omega. \tag{1}$$

If Ω is simply connected and $u \in W^{1,2}_{loc}(\Omega)$ solves the equation $\text{div}(\sigma\nabla u) = 0$, let $v \in W^{1,2}_{loc}(\Omega)$ be its conjugate, with $\nabla v = *\sigma(x)\nabla u$. Show that

$$\text{div}(\sigma^*(x)\nabla v) = 0 \text{ in } \Omega, \quad \text{where } \sigma^* = - * \sigma^{-1}(x) * .$$

Show also that σ^* satisfies the same ellipticity bounds (1), and that if $\det \sigma(x) \equiv 1$ and σ is symmetric, i.e. $\sigma(x)^t = \sigma(x)$, then $\sigma^* = \sigma$.

5. Suppose $\sigma : \Omega \to \mathbb{R}^{2\times 2}$ is symmetric and $\det \sigma(x) \equiv 1$, with Ω simply connected.

Show directly, without referring to Theorem 5.5, that $u \in W^{1,2}_{loc}(\Omega)$ satisfies the equation $\operatorname{div}(\sigma \nabla u) = 0$ if and only if $\overline{\partial} f = \mu \, \partial f$, where $f = u + iv$, v is the conjugate of u and

$$\mu = \frac{\sigma_{22} - \sigma_{11} - 2i\sigma_{12}}{2 + \operatorname{Tr}(\sigma)} \, .$$