Department of Mathematics and Statistics
Quasiconformal mappings and elliptic PDE's
Exercise set 1.
5.2.2013

1. a) Suppose the function $f : \Omega \to \mathbb{C}$ is differentiable at point $z \in \Omega$.

Show that with the complex derivatives $\partial f(z)$ and $\overline{\partial} f(z)$ the derivative gets the form

$$Df(z)(h) = \partial f(z)h + \overline{\partial} f(z)\overline{h}$$

b) Show that for any mapping $f : \Omega \to \mathbb{C}$ the derivatives of f and $\overline{f}$ are related by $\overline{\partial f(z)} = \overline{\partial}\,\overline{f}(z)$, i.e. that $\overline{f_z(z)} = \overline{f}_{\overline{z}}(z)$ at every point $z \in \Omega$ where f is differentiable. Show also that the Jacobian determinant $J_f(z) \equiv \det Df(z) = |\partial f(z)|^2 - |\overline{\partial} f(z)|^2$.

c) Show that if g is differentiable at z and f is differentiable at $g(z)$, then the chain rule obtains the form

$$\overline{\partial}(f \circ g)(z) = (\partial f)(gz)\overline{\partial} g(z) + (\overline{\partial} f)(gz)\overline{\partial g(z)} = (\partial f)(gz)\overline{\partial} g(z) + (\overline{\partial} f)(gz)\overline{\partial}\,\overline{g}(z)$$

$$\partial(f \circ g)(z) = (\partial f)(gz)\partial g(z) + (\overline{\partial} f)(gz)\overline{\overline{\partial} g(z)} = (\partial f)(gz)\partial g(z) + (\overline{\partial} f)(gz)\partial \overline{g}(z)$$

2. Show that a map $f : \mathbb{C} \to \mathbb{C}$ is a similarity $\Leftrightarrow$

$$\frac{f(z) - f(w)}{f(z) - f(\zeta)} = \frac{z - w}{z - \zeta} \qquad \text{for all distinct } z, w, \zeta \in \mathbb{C}.$$

3. Let $f(z) = |z|^\alpha$, $z \in \mathbb{C}$ and $\alpha > -1$. If $g(z) = \frac{\alpha}{2}\,\overline{z}\,|z|^{\alpha-2}$, show that $\int_\Omega f(z)\partial\phi(z)dm(z) = -\int_\Omega g(z)\phi(z)dm(z)$ for every $\phi \in C_0^\infty(\mathbb{C})$. Conclude that the weak derivative $\partial f(z) = \frac{\alpha}{2}\,\overline{z}\,|z|^{\alpha-2}$.

[Hint: supp $(\phi) \subset \Omega$, denote $\Omega_\varepsilon = \Omega \backslash B(0,\varepsilon)$. Use Green's formula $\int_{\partial\Omega_\varepsilon} h d\overline{z} = (-1/2i)\int_{\Omega_\varepsilon} \partial h\, dm$ from Theorem AII.1 in Appendix II, and let $\varepsilon \to 0$.]

4. Suppose f is η-quasisymmetric in a domain $\Omega \subset \mathbb{C}$. For a disk $B = B(z_0, r)$ write $sB := B(z_0, sr)$.

Show that the measure $\nu(A) = |f(A \cap \Omega)|$ is doubling on disks contained in Ω, i.e.

$$|f(2B)| \leq C(\eta)|f(B)| \qquad \text{when } 2B \subset \Omega$$

5. Suppose $A \subset \mathbb{C}$ with $0, 1 \in A$.

a) Show that the family of all $\eta-$quasisymmetric mappings f on A with $f(0) = 0$ and $f(1) = 1$ is equicontinuous and pointwise bounded. Here η is fixed.

b) By Ascoli-Arzela theorem and a), every sequence (f_n) of $\eta-$quasisymmetric maps contains a locally uniformly converging subsequence (f_{n_k}). Show that the limit $f = \lim_k f_{n_k}$ is $\eta-$quasisymmetric on A.

6. Suppose that F is an $\eta-$quasisymmetric mapping defined in $\mathbb{C}$, and with range $F(\mathbb{C}) \subset \mathbb{C}$. Show that necessarily $F(\mathbb{C}) = \mathbb{C}$.

Note: Combined with Theorem 3.22 this shows that quasiconformal maps defined in $\mathbb{C}$ are surjections onto $\mathbb{C}$.