GERBES, (TWISTED) K-THEORY, AND THE SUPERSYMMETRIC WZW MODEL

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ABSTRACT The aim of this talk is to explain how symmetry breaking in a quantum field theory problem leads to a study of projective bundles, Dixmier-Douady classes, and associated gerbes. A gerbe manifests itself in different equivalent ways. Besides the cohomological description as a DD class, it can be defined in terms of a family of local line bundles or as a prolongation problem for an (infinite-dimensional) principal bundle, with the fiber consisting of (a subgroup of) projective unitaries in a Hilbert space. The prolongation aspect is directly related to the appearance of central extensions of (broken) symmetry groups. We also discuss the construction of twisted K-theory classes by families of supercharges for the supersymmetric Wess-Zumino-Witten model.

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0. INTRODUCTION

In quantum field theory gerbes arise when one asks the question whether a given bundle of quantum mechanical projective spaces is a projectivization of a Hilbert space bundle. Nontrivial obstructions to the existence of the Hilbert bundle are generated by QFT anomalies.

An anomaly in field theory is a breakdown of a group of (gauge) symmetries in the quantization of a classical field theory model. A classical symmetry can be broken in the quantization of massless fermions in a classical background field. The background consists typically of a curved space-time metric or a Yang-Mills field. Because of the breakdown of the symmetry, in the quantized theory one cannot identify gauge or diffeomorphism equivalent Hilbert spaces. The quantized symmetry group acts only projectivly, through a central (or an abelian) extension; for this reason modding out by the symmetry group leads to a bundle of projective spaces parametrized by the external field configurations.

The obstruction to replacing a projective bundle over X by a true vector bundle is given by a cohomology class in $H^3(X,\mathbb{Z})$, the *Dixmier-Douady class.* At the same time, the Dixmier-Douady class describes a (stable) equivalence class of a gerbe over X. For the general theory of gerbes, the reader is recommended to consult [Br]. Here our discussion is closely related to a specialized form, the *bundle gerbe,* introduced in [Mu], which is an abstraction of a quantum field theory problem involving massless fermions, [Mi].

A gerbe over X can be viewed as a collection of local line bundles L_{ij} over intersections $U_{ij} = U_i \cap U_j$ of open subsets of X. In addition, the gerbe data involves a family of isomorphisms

$$L_{ij} \otimes L_{jk} = L_{ik}$$

on the triple overlaps U_{ijk}. Given a family of curvature forms ω_{ij} for the local line bundles, satisfying the cocycle property $\omega_{ij} + \omega_{jk} = \omega_{ik}$, one can produce a closed integral 3-form ω defined on X using a standard construction in a Cech - de Rham double complex. The class $[\omega] \in H^3(X,\mathbb{Z})$ is de Rham form of the Dixmier-Douady class of the gerbe. (This is not the whole story, because there are cases when the DD class is pure torsion.)

The local line bundles L_{ij} arise in a natural way when trying to deprojectivisize a projective bundle over X. The transition functions on the overlaps U_{ij} of a projective bundle are given as functions g_{ij} taking values in a projective unitary group. The unitary group $U(H)$ is a central extension by the circle S^1 of the projective unitary group $PU(H)$. Thus the possible unitaries $\hat{g}_{ij}(x)$ representing $g_{ij}(x) \in PU(H)$ form a circle over the point $x \in U_{ij}$; replacing the circle by $\mathbb{C}$ we obtain a complex line $L_{ij}(x)$. The group product in $U(H)$ gives a natural identification of $L_{ij} \otimes L_{jk}$ as L_{ik} in the common domain.

Sections 2 and 3 contain an introduction to the (twisted) K-theory aspects of gerbes, from the quantum field theory point of view. K-theory arises naturally in (hamiltonian) quantization. We have a family of Hamilton operators parametrized by points in X. When the Hamilton operators are self-adjoint and have both positive and negative essential spectrum, they give (by definition) an element of $K^1(X)$. It is known that $K^1(X)$ is parametrized by the odd cohomology groups $H^{2k+1}(X,\mathbb{Z})$. The Dixmier-Douady class is then the projection to the 3-cohomology part. If we disregard torsion, we can describe this by a de Rham form of degree 3. This is also the starting point for constructing the twisted K-theory classes, [Ro]. In section 4 we give an explicit example using the supersymmetric WZW model.

Of course, in this short presentation I have left out many interesting topics; for recent discussions on the applications of gerbes and K-theory to strings and conformal field theory see e.g. [GR], [Se].

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1. GERBES FROM CANONICAL QUANTIZATION

Let us first recall some basic facts about canonical anticommutation relations (CAR) in infinite dimensions, for an extensive review see [Ar]. Let H be a complex Hilbert space. To each vector $u \in H$ one associates a pair of elements $a(u)$ and $a^*(u)$ which are generators of a complex unital algebra, the CAR algebra based on H. The basic relations are

$$\begin{aligned} &a(u)a^*(v) + a^*(v)a(u) = <u, v> \mathbf{1} \\ &a(u)a(v) + a(v)a(u) = 0 = a^*(u)a^*(v) + a^*(v)a^*(u) \end{aligned} \tag{1.1}$$

for all vectors $u, v \in H$. The map $u \mapsto a^*(u)$ is linear whereas $u \mapsto a(u)$ is antilinear.

Each polarization $H = H_+ \oplus H_-$ to a pair of infinite-dimensional subspaces defines an irreducible representation of CAR in a dense domain of a Hilbert space (the *Fock space*) $\mathcal{F} = \mathcal{F}(H_+ \oplus H_-)$. The (equivalence class of the) representation is uniquely defined by the requirement of existence of a *vacuum vector* ψ_0 such that

$$a^*(u)\psi_0 = 0 = a(v)\psi_0 \tag{1.2}$$

for all $u \in H_-$ and $v \in H_+$. Any two representations defined by polarizations H_+, H'_+ are equivalent if and only if the projection operators P, P' to these subspaces differ by a Hilbert-Schmidt operator.

A unitary operator $S : H \to H$ can be promoted to a unitary operator $\hat{S} : \mathcal{F} \to \mathcal{F}$ such that

$$\hat{S}a(u)\hat{S}^{-1} = a(Su) \ \ \forall u \in H,$$

and similarly for the creation operators $a^*(u)$, if and only if the off-diagonal blocks of S (in the given polarization) are Hilbert-Schmidt operators. Let us denote the group of unitaries S of this type as $U_{res} = U_{res}(H_+ \oplus H_-)$, [PS]. Note that the operator $\hat{S}$ is only defined up to a phase factor. The group of quantum operators $\hat{S}$ forms a central extension $\hat{U}_{res}$ of U_{res},

$$1 \to S^1 \to \hat{U}_{res} \to U_{res} \to 1.$$

Likewise, a bounded linear operator $X : H \to H$ can be 'second quantized' as a linear operator $\hat{X}$ in $\mathcal{F}$ such that

$$[\hat{X}, a^*(u)] = a^*(Xu), \ \ [\hat{X}, a(u)] = -a(X^*u) \tag{1.3}$$

for all $u \in H$ if and only if the off-diagonal blocks of X are Hilbert-Schmidt. In this case the operator $\hat{X}$ is uniquely defined modulo an additive constant. The operators $\hat{X}$ form a Lie algebra, a central extension of the Lie algebra of the complex group GL_{res}, with commutation relations

$$[\hat{X}, \hat{Y}] = \widehat{[X, Y]} + c(X, Y),$$

where the complex valued 2-cocycle c depends on certain choices (physically, the choice of normal ordering in the Fock space), but its cohomology class is represented by, [Lu],

$$c(X, Y) = \frac{1}{4} \mathrm{tr}\, \epsilon[\epsilon, X][\epsilon, Y],$$

where $\epsilon = P_{H_+} - P_{H_-}$.

In quantum field theory problems the polarization arises as a splitting of the 1-particle Hilbert space into positive and negative energy subspaces with respect to a (in general unbounded) self adjoint Hamilton operator (e.g. a Dirac operator).

Consider next a case when we have a parametrized family of Hamilton operators. Let X be some manifold and let for each $x \in X$ a self adjoint operator D_x (in a dense domain of) H be given, such that D_x depends smoothly on the parameter x in some appropriate topology. Tentativly, we would like to construct a family of Fock spaces $\mathcal{F}_x$ defined by the polarizations $H = H_+(D_x) \oplus H_-(D_x)$ to positive and negative spectral subspaces. However, in general this is not possible in a smooth way because of the spectral flow: each time an eigenvalue $\lambda_n(x)$ of D_x crosses the zero mode $\lambda = 0$ we have a discontinuity in the polarization and thus in the construction of the Fock spaces.

The potential resolution to the above problem lies in the fact that one is really interested only in the equivalence class of the CAR representation. Therefore one is happy with a choice of a function $x \mapsto P_x$, where P_x is a projection operator which differs from the projection operator onto $H_+(D_x)$ by a Hilbert-Schmidt operator. However, there can be an obstruction to the existence of the function P_x which depends on the K-theory class of the mapping $x \mapsto D_x$.

Recall the operator theoretic meaning of $K^1(X)$. Let $Fred_*$ be the space of self-adjoint Fredholm operators in H with both negative and positive essential spectrum. Then $K^1(X)$ can be identified as the space of homotopy classes of maps $X \to Fred_*$. In particular, a family D_x of Dirac type operators defines an element in $K^1(X)$. Up to torsion, $K^1(X)$ is parametrized by the odd de Rham cohomology classes in $H^*(X, \mathbb{Z})$. It turns out that for the existence of the family P_x only the 3-form part is relevant.

As a concrete example in quantum field theory consider the case of Dirac operators coupled to vector potentials. Let $\mathcal{A}$ be the space of $\mathbf{g}$ valued 1-forms on a compact odd dimensional spin manifold M where $\mathbf{g}$ is the Lie algebra of a compact group G. Each $A \in \mathcal{A}$ defines a Dirac operator D_A in the space H of square integrable spinor fields twisted by a representation of G. The 'free' Dirac operator D_0 defines a background polarization $H = H_+ \oplus H_-$. Each potential A and a real number λ defines a spectral subspace $H_+(A,\lambda)$ corresponding to eigenvalues of D_A strictly bigger than λ.

Let $Gr_p(H_+)$ be the Grassmann manifold consisting of all closed subspaces $W \subset H$ such that $P_W - P_{H_+}$ is in the Schatten ideal L_p of operators T with $|T|^p$ a trace-class operator. One can show that each $H_+(A,\lambda)$ belongs to $Gr_p(H_+)$ when $p > \dim M$. All the Grassmannians Gr_p for $p \geq 1$ are homotopy equivalent and they are also homotopy equivalent to the space $Fred$ of all Fredholm operators in H. For this reason Gr_p is a classifying space in K-theory. In particular, the connected components of Gr_p are labelled by the Fredholm index of the projection $W \to H_+$ and each component is simply connected. The second integral cohomology (and second homotopy) of each component is equal to $\mathbb{Z}$. For this reason the complex line bundles are generated by a single element DET_p. In the case $p \leq 2$ the curvature of the dual line bundle DET_p^* is particularly simple; it is given as 2π times the normalized 2-form

$$\omega = \frac{1}{16\pi} \mathrm{tr}\, PdPdP. \tag{1.4}$$

We can cover $\mathcal{A}$ with the open sets $U_\lambda = \{A \in \mathcal{A} | \lambda \notin Spec(D_A)\}$. On each U_λ the map $A \mapsto H_+(A,\lambda) \in Gr_p$ is smooth. For this reason we may pull back the line bundle DET_p to a line bundle $DET_{p,\lambda}$ over U_λ. We shall not go into the explicit construction of DET_p here, [MR]. Instead, the difference bundles $DET_{p,\lambda\lambda'} = DET_{p,\lambda} \otimes DET^*_{p,\lambda'}$ over $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$ are easy to describe. The fiber of $DET_{p,\lambda\lambda'}$ is simply the top exterior power of the spectral subspace of D_A for $\lambda < D_A < \lambda'$. By construction, we have a canonical identification

$$DET_{p,\lambda\lambda'} \otimes DET_{p,\lambda'\lambda''} = DET_{p,\lambda\lambda''} \tag{1.5}$$

on the triple overlaps, for $\lambda < \lambda' < \lambda''$. We set $DET_{p,\lambda\lambda'} = DET^*_{p,\lambda'\lambda}$ for $\lambda > \lambda'$. A system of complex line bundles with the cocycle property (1.5) defines a a *gerbe.* In this case we have a trivial gerbe, it is generated by local line bundles $DET_{p,\lambda}$ over the open sets U_λ. However, we may push things down to the space of gauge orbits $X = \mathcal{A}/\mathcal{G}$, where $\mathcal{G}$ is the group of *based gauge transformations,* i.e., the group of smooth maps $f : M \to G$ such that $f(p) = 1$ at a base point $p \in M$; here we assume for simplicity that M is connected.

The gauge group acts covariantly on $\mathcal{A}$ and on the eigenvectors of D_A and therefore we may mod out by the $\mathcal{G}$ action to manufacture complex line bundles over $V_{\lambda\lambda'} = U_{\lambda\lambda'}/\mathcal{G}$. These line bundles (which we also denote by $DET_{p,\lambda\lambda'}$) satisfy the same cocycle property (5) as the original bundles over $U_{\lambda\lambda'}$. Thus we obtain a gerbe over X. Generically, this gerbe is nontrivial. In general, there is an obstruction to the trivialization of the gerbe given as a *Dixmier-Douady class* in $H^3(X,\mathbb{Z})$.

The Dixmier-Douady class can be computed as follows. Let $\omega_{\lambda\lambda'}$ be the curvature form of $DET_{\lambda\lambda'}$. These satisfy

$$\omega_{\lambda\lambda'} + \omega_{\lambda'\lambda''} = \omega_{\lambda\lambda''} \tag{1.6}$$

on the triple overlaps. Let $\{\rho_\lambda\}$ be a partition of unity subordinate to the covering by the open sets V_λ. The closed 3-forms

$$\omega_\lambda = \sum_{\lambda'} d\rho_{\lambda'}\omega_{\lambda\lambda'}$$

satisfy $\omega_\lambda = \omega_{\lambda'}$ on $V_{\lambda\lambda'}$ and therefore can be glued together to produce a global closed 3-form ω. This is easily seen to be integral. The class $[\omega] \in H^3(X,\mathbb{Z})$ is the DD class of the gerbe.

Let us again consider the case of the trivial gerbe over $\mathcal{A}$. The trivialization by the local line bundles $DET_{p,\lambda}$ resolves the problem of defining a continuous family of CAR representations parametrized by $\mathcal{A}$. Over each set U_λ we can define a CAR algebra representation in $\mathcal{F}'(A,\lambda) = \mathcal{F}(A,\lambda) \otimes DET^*_{p,\lambda}$ by the action $a^*(u) \otimes 1$ and $a(u) \otimes 1$ on the fibers. The crux is that the spaces $\mathcal{F}'(A,\lambda)$ are canonically isomorphic for different values of λ, and hence we have a well-defined family of spaces $\mathcal{F}'(A)$ for all $A \in \mathcal{A}$, [Mi]. In physics terminology, an isomorphism between $\mathcal{F}(A,\lambda)$ and $\mathcal{F}(A,\lambda')$ for $\lambda < \lambda'$ is obtained by 'filling the Dirac sea' from the vacuum level λ up to the level λ'. The filling is canonically defined up to a unitary rotation of the eigenvectors of D_A in the spectral interval $[\lambda,\lambda']$; a rotation of the basis by R leads to a phase factor $\det R$ in the filling, which is exactly compensated by the inverse phase factor in the isomorphism between the dual determinant lines $DET^*_{p,\lambda}(A)$ and $DET^*_{p,\lambda'}(A)$.

2. K-THEORY ASPECTS OF CANONICAL QUANTIZATION

Let X be a parameter space for a family of self-adjoint operators with both positive and negative essential spectrum, that is, we have a map $X \to Fred_*$ or in other words we have an element in $K^1(X)$.

As shown in [AS] the space $Fred_*$ is homotopy equivalent to the group of unitaries g in the complex Hilbert space H such that $g-1$ is compact. According to Palais this group is homotopy equivalent to the group $U^{(p)}$ of unitaries g such that $g-1 \in L_p$ for any $p \geq 1$. The choice $G = U^{(1)}$ is the most convenient one since it allows to write in a simple way the generators for $H^*(G,\mathbb{Z})$. The cohomology is generated by the odd elements

$$c_{2k+1} = N_k \mathrm{tr}\,(dgg^{-1})^{2k+1}$$

for $k = 0,1,2,\ldots$; $N_k = -(1/2\pi i)^{k+1}\frac{k!}{(2k+1)!}$ is a normalization constant.

The infinite-dimensional group U_{res} is of interest to us. Let $\mathcal{H} = L^2(S^1,H)$. Then the group ΩG of smooth based loops in G acts naturally in $\mathcal{H}$. The space $\mathcal{H}$ has a natural polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ to positive (resp. zero and negative) Fourier modes. The ΩG orbit of $\mathcal{H}_+$ lies in the Hilbert-Schmidt Grassmannian

$Gr_2(\mathcal{H}_+)$. Thus we have $\Omega G \subset U_{res}(\mathcal{H}_+ \oplus \mathcal{H}_-)$. Actually, the inclusion is a homotopy equivalence (a consequence of Bott periodicity), [CM1].

There is a universal ΩG bundle P over G, with total space the set of smooth paths $f : [0,1] \to G$ starting from the unit element and such that $f^{-1}df$ is periodic at the end points. Replacing ΩG by U_{res} we obtain an universal U_{res} bundle over G. Thus G is a classifying space for U_{res} bundles and we have:

Proposition. *$K^1(X)$ is isomorphic to the group of equivalence classes of U_{res} bundles over X.*

The group structure in $K^1(X)$ comes from the representation of elements in $K^1(X)$ as homotopy classes of maps $g : X \to G$. The product is the pointwise multiplication of maps.

In canonical quantization, it is the U_{res} bundle aspect of $K^1(X)$ which is seen more directly. As we discussed earlier, the Fock representations of the CAR algebra are determined by polarizations of the 1-particle space. A family D_x of self-adjoint operators in $Fred_*$ defines a principal U_{res} bundle over X. The fiber of the bundle at x is the set of unitaries g in $H = H_+ \oplus H_-$ such that the projection onto gH_+ differs from the spectral projection to the subspace $D_x > 0$ by a Hilbert-Schmidt operator. Since $Fred_* \simeq G$ this bundle is the pull-back of the universal bundle P over G by the map $x \mapsto D_x$.

The second quantization may be viewed as a prolongation problem for a U_{res} bundle P_D over X. We want to construct a vector bundle $\mathcal{F}$ over X such that the fibers are Fock spaces carrying representations of the CAR algebra. The Fock bundle is an associated bundle, not to P_D because U_{res} does not act in the Fock spaces, but to an $\hat{U}_{res}$ bundle $\hat{P}_D$ which is a prolongation of P_D by the center S^1 of $\hat{U}_{res}$. The following was proven in [CM1]:

Theorem 1. *There is an obstruction for prolonging P_D to $\hat{P}_D$ given by the Dixmier-Douady class which is in the 3-form part of the K-theory class $[D] \in K^1(X)$.*

In many cases the DD class can be computed using index theory, see [CMM1,2], [CM1,2], but here we shall discuss a bit more the calculation based on the homotopy equivalence $Fred_* \simeq G$.

First, one can construct a homotopy equivalence from $Fred_*$ to the space of *bounded* self-adjoint operators with essential spectrum at ± 1, [AS]. For Dirac type operators on a compact manifold we could take for example the map $D \mapsto F_D = D/(|D| + \exp(-D^2))$. This is the approximate sign operator of D. The next step is to map the F_D's to unitary operators by $F_D \mapsto g_D = -\exp(i\pi F_D)$. It is not difficult to see that the difference $g_D - 1$ is trace-class. Note that for Dirac operators on a compact manifold of dimension n we could take $F_D = D/(|D| + 1)$, for simplicity, but then we would have the weaker condition $g_D - 1 \in L_p$ for $p > n$.

Formally, the pull-back of the universal DD class

$$c_3 = \frac{1}{24\pi^2} \mathrm{tr}\,(dgg^{-1})^3$$

with respect to the mapping $F \mapsto g = -\exp(i\pi F)$ can be computed as $\omega = d\theta$ with

$$\theta = \frac{1}{8} \mathrm{tr}\, dF h(ad_{i\pi F}) dF \tag{2.1}$$

where $h(x) = (\sinh(x) - x)/x^2$ and $ad_F(z) = [F, z]$. However, in general the expression after 'tr' is not trace-class (and for this reason ω is not exact). There are interesting cases when the above formula makes sense. When D is a Dirac operator on a manifold of dimension n then one can show using standard estimates on pseudodifferential symbols that dF is in L_p for any $p > n$. It follows that the right-hand-side of (2.1) is well-defined for a Dirac operator on a circle. The case of dimension $n = 3$ is a limiting case. In three dimensions the trace is logarithmically divergent (when defined as a conditional trace in a basis where F is diagonal) and subtracting the logarithmic divergence one obtains a finite expression which can be used to define θ.

The case $F^2 = 1$ is also interesting; these points define a Grassmann manifold since $(F+1)/2$ is a projection onto an infinite-dimensional subspace. Again, in the case of sign operators defined by Dirac operators one can show that F defines a point in Gr_p for $p > n$. Assuming that the trace in (2.1) converges, one obtains a specially simple formula for the form θ,

$$\theta = \frac{1}{16\pi} tr\, F dF dF \text{ for } F^2 = 1 \ .$$

Not surprisingly, this is (mod a factor 2π) the curvature formula for the determinant bundle DET_2 over Gr_2.

3. TWISTED K-THEORY AND QFT

There has been an extensive discussion of twisted K-theory in the recent string theory literature, inspired by suggestions in [Wi]. I will not discuss any of the string theory applications here. Instead, I want to point out that twistings in K-theory are related to some very basic constructions in standard QFT.

Twisted K-theory was introduced in [Ro] as a generalization of algebraic K-theory on C^*-algebras. Today there are several equivalent definitions of twisted K, see e.g. [BCMMS]. For QFT problems I find it most convenient to use the topologist definition of twisted K-theory groups.

Twisted K-theory elements arise from principal $PU(H)$ bundles. Here $PU(H) = U(H)/S^1$ is the projective unitary group in a complex Hilbert space. Since $U(H)$ is contractible by Kuiper's theorem, the homotopy type of $PU(H)$ is simple: The only nonzero homotopy group is $\pi_2(PU(H)) = \mathbb{Z}$. For this reason $H^2(PU(H), \mathbb{Z}) = \mathbb{Z}$. On the Lie algebra level, the basic central extension $1 \to S^1 \to U(H) \to PU(H) \to 1$ of $PU(H)$ is given as follows. Let $\psi_0 \in H$ be a fixed vector of unit length and $\mathbf{g}_0$ the subspace in the Lie algebra $\hat{\mathbf{g}}$ of $U(H)$ consisting of operators x such that

$(\psi_0, x\psi_0) = 0$. For each y in the Lie algebra $\mathbf{g} = \hat{\mathbf{g}}/i\mathbb{R}$ of $PU(H)$ there is a unique element $\hat{y} \in \mathbf{g}_0$ such that $\pi(\hat{y}) = y$, where π is the canonical projection. We can write $\hat{\mathbf{g}} = \mathbf{g} \oplus i\mathbb{R}$ and the commutation relations in $\hat{\mathbf{g}}$ can be written as

$$[(x,\alpha),(y,\beta)] = ([x,y]_{\mathbf{g}}, c(x,y))$$

where the cocycle c is given by $c(x,y) = [\hat{x},\hat{y}] - \widehat{[x,y]}_{\mathbf{g}}$.

Given a principal $PU(H)$ bundle P over X we we can construct an associated $Fred_*$ bundle $Q(P)$ over X as $P \times_{PU(H)} Fred_*$ where $PU(H)$ acts on $Fred_*$ through conjugation. Again, using the homotopy equivalence $Fred_* \simeq G$ we might consider G bundles as well; but then it is important to keep in mind that these are not principal G bundles. The twisted K^1 of X, to be denoted by $K^1(X,[P])$, is then defined as the set of homotopy classes of sections of the bundle $Q(P)$. A similar definition is used for the twisted K^0 group, the space $Fred_*$ is then replaced by the space of all Fredholm operators in H, or alternatively, we can use the model $U_{res} \simeq Fred$.

Quantum field theory provides concrete examples of twisted bundles $Q(P)$ and its sections. As we have seen, a family of Dirac type hamiltonians parametrized by X is an element of $K^1(X)$ or equivalently, an equivalence class of U_{res} bundles over X. The basic observation is that the group U_{res} can be viewed as a subgroup of $PU(\mathcal{F})$ where $\mathcal{F}$ is a Fock space carrying a representation of the central extension $\hat{U}_{res}$. Therefore we can extend the U_{res} bundle to a $PU(\mathcal{F})$ bundle P over X. A section of the associated bundle $Q(P)$ becomes now a function f from P to the space of operators of type $Fred_*$ in the Fock space $\mathcal{F}$ such that $f(pg) = g^{-1}f(p)g$ for all $g \in PU(\mathcal{F})$ and $p \in P$. Since our $PU(\mathcal{F})$ reduces to a U_{res} bundle P_0 we might as well construct a section of $Q(P)$ from an equivariant function on P_0 with values in $Fred_*$.

Let U_i be a family of open sets covering X equipped with local trivializations of the U_{res} bundle. Let $g_{ij} : U_i \cap U_j \to U_{res}$ be the corresponding transition functions. Then a section of $Q(P)$ can be given by a family of maps

$$a_i : U_i \to Fred_*(\mathcal{F})$$

such that

$$a_j(x) = \hat{g}_{ij}(x)^{-1} a_i(x) \hat{g}_{ij}(x) \quad \text{for } x \in U_{ij} \ .$$

Here $\hat{g}$ is the quantum operator, acting in $\mathcal{F}$, corresponding to the '1-particle operator' g.

In general, twisted K^1 is not easy to compute. Let us consider a simple example.

Example Let $X = S^3$ which can be identified as the group $SU(2)$ of unitary 2×2 matrices of determinant $= 1$. A twisted $PU(H)$ bundle P over S^3 is constructed as follows. First, the space of smooth vector potentials A on a circle with values in the Lie algebra of $SU(2)$ can be viewed as a principal $\Omega SU(2)$ bundle over S^3. The group of based loops $\Omega SU(2)$ acts on the vector potentials through gauge transformations

$A \mapsto g^{-1}Ag + g^{-1}dg$. A vector potential modulo based gauge transformations is parametrized by the holonomy around the circle, giving an element in $S^3 = SU(2)$.

The loop group $\Omega SU(2)$ acts in the space H of square-integrable $\mathbb{C}^2$ valued functions on the unit circle, giving an embedding $\Omega SU(2) \subset U_{res}(H_+ \oplus H_-)$, the polarization being given by the splitting to negative and nonnegative Fourier modes. Thus we can extend the $\Omega SU(2)$ bundle $\mathcal{A}$ over S^3 to a principal U_{res} bundle P_0. This extends, as explained above, to a principal $PU(\mathcal{F})$ bundle over S^3.

All principal $PU(\mathcal{F})$ bundles over S^3 are classified by the homotopy classes of transition functions $S^2 \to PU(\mathcal{F})$, that is, by the the elements $n \in \pi_2(PU(\mathcal{F})) = \mathbb{Z}$. The construction above gives the basic bundle with $n = 1$. The higher bundles are obtained by taking tensor powers (and their duals) of the Fock space representations of the central extension of the loop group $\Omega SU(2)$. The K^1-theory twist in this case is fixed by a choice of the integer n. An element of the twisted $K^1(S^3, n)$ is now given by the homotopy class of pairs of functions $h_\pm : S^3_\pm \to Fred_*$ such that on the equator $S^2 \sim S^3_+ \cap S^3_-$

$$h_+(x) = g^{-1}(x)h_-(x)g(x),$$

where $g : S^2 \to PU(H)$ is can be given explicitly, using the embedding $\Omega SU(2) \subset PU(\mathcal{F})$ and the fact that $\pi_2 \Omega SU(2) = \mathbb{Z}$, see [CMM2] for details.

All classes in $K^1(SU(2), n)$ can actually be given in a simpler way. We can use the homotopy equivalence $Fred_* \simeq U^{(1)} = G$. Choose then $h_+ : S^3_+ \to G$ such that it is equal to the unit element on the overlap $S^3_+ \cap S^3_-$ and take h_- as the constant function on S^3_- taking the value $1 \in G$. Then clearly $h_\pm$ are intertwined by the transition function g on the overlap. Since h_+ is constant on the boundary of S^3_+, it can be viewed as a map $g_+ : S^3 \to G$. The winding number of this map in $\pi_3(G) = \mathbb{Z}$, modulo n, determines the class in $K^1(S^3, n) = \mathbb{Z}/n\mathbb{Z}$.

The transition function in the present example, being a map from S^2 to U_{res} (which is a classifying space for K^0), can also be viewed as an element of $K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$. This is an essential part of the computation of $K^1(S^3, n)$, based on the Mayer-Vietoris theorem in K-theory, see [BCMMS] for details, or the original computation in [Ro].

4. AN EXAMPLE: SUPERSYMMETRIC WZW MODEL

Let H_b be a complex Hilbert space carrying an irreducible unitary highest weight representation of the central extension $\widehat{LG}$ of the loop group LG of level k; here G is assumed to be compact and simple, $\dim G = N$. The level satisfies $2k/\theta^2 = 0, 1, 2, \dots$, where θ is the length of the longest root of G.

Let H_f be a fermionic Fock space for the CAR algebra generated by elements ψ^a_n with $n \in \mathbb{Z}$ and $a = 1, 2, \dots, N = \dim G$,

$$\psi^a_n \psi^b_m + \psi^b_m \psi^a_n = 2\delta_{n,-m}\delta_{a,b}. \tag{4.1}$$

The Fock vacuum is a subspace of H_f of dimension $2^{[N/2]}$ (here $[p]$ denotes the integral part of a real number p). The vacuum subspace carries an irreducible representation of the Clifford algebra generated by the ψ_0^a's and in addition any vector in the vacuum subspace is annihilated by all ψ_n^a's with $n < 0$.

The tensor product space $H = H_f \otimes H_b$ carries a tensor product representation of $\widehat{LG}$. The fermionic part of the representation is determined by the requirement

$$T_f(g)\psi(\alpha)T_f(g)^{-1} = \psi(g \cdot \alpha), \tag{4.2}$$

where α is a $\mathbb{C}^N$ valued smooth function on the unit circle and $\psi(\alpha) = \sum \psi_n^a \alpha_{-n}^a$, where the α_n^a's are the Fourier components of the vector valued function α. The action of $g \in LG$ on α is the point-wise adjoint action on the Lie algebra of the loop group.

The Lie algebra of $\widehat{LG}$ acting in H_b is generated by the Fourier modes T_n^a subject to the commutation relations

$$[T_n^a, T_m^b] = \lambda_{abc} T_{n+m}^c + kn\delta_{n,-m}\delta_{a,b}, \tag{4.3}$$

where the λ_{abc}'s are the structure constants of the Lie algebra $\mathbf{g}$ in a basis T^a which is orthonormal with respect to the Killing form $< X, Y >= -\mathrm{tr}(ad_X \cdot ad_Y)$. There is, up to a phase factor, a unique normalized vector $x_\lambda \in H_b$ such that $T_n^a x_\lambda = 0$ for $n < 0$ and is a highest weight vector of weight λ for the finite-dimensional Lie algebra $\mathbf{g}$.

We denote the loop algebra generators acting in the fermionic Fock space H_f by K_n^a. They satisfy the commutation relations

$$[K_n^a, K_m^b] = -\lambda_{abc} K_{n+m}^c + \frac{1}{2} n\delta_{n,-m}\delta_{a,b}. \tag{4.4}$$

Explicitly, the generators are given by

$$K_n^a = -\frac{1}{4}\lambda_{abc} : \psi_{n+m}^b \psi_{-m}^c : . \tag{4.5}$$

The normal ordering :: is defined as the rule to write the operators with negative momentum index to the right of those with positive index. Actually, since our λ_{abc}'s are totally antisymmetric, the normal ordering in (4.5) is irrelevant.

We denote by S_n^a the generators of the tensor product representation in $H = H_b \otimes H_f$. They satisfy the relations

$$[S_n^a, S_m^b] = \lambda_{abc} S_{n+m}^c + (k + \frac{1}{2}) n\delta_{ab}\delta_{n,-m}. \tag{4.6}$$

The free hamilton operator is

$$h = h_b \otimes 1 + 1 \otimes (2k+1)h_f + \frac{N}{24}(1 \otimes 1)$$

where

$$h_b = - : T^a_n T^a_{-n} : \text{ and } h_f = -\frac{1}{4} : n\psi^a_n \psi^a_{-n} : \tag{4.7}$$

We use the conventions

$$(\psi^a_n)^* = \psi^a_{-n} \text{ and } (T^a_n)^* = -T^a_{-n}. \tag{4.8}$$

As seen by a direct computation, the supercharge Q satisfies $Q^2 = h$ and is defined by

$$Q = i\psi^a_{-n} T^a_n - \frac{i}{12}\lambda_{abc}\psi^a_n \psi^b_m \psi^c_{-m-n}. \tag{4.9}$$

For a detailed description of the whole super current algebra, see [KT], or in somewhat different language, [La]. Again, by antisymmetry of the structure constants, no normal ordering is necessary in the last term on the right. The general structure of Q has similarities with Kostant's cubic Dirac operator, [Ko], (containing a cubic term in the 'gamma matrices' ψ^a_n); another variant of this operator has been discussed in conformal field theory context in [B-R]. Restricting to zero momentum modes, the operator Q in fact becomes Kostant's operator

$$\mathcal{K} = i\gamma^a T^a - \frac{i}{12}\lambda_{abc}\gamma^a\gamma^b\gamma^c, \tag{4.10}$$

where $\psi^a_0 = \gamma^a$ are the Euclidean gamma matrices in dimension N. By the antisymmetry of the structure constants the last term is totally antisymmetrized product of gamma matrices.

The supercharge is a hermitean operator in a dense domain of the Fock space H, including all the states which are finite linear combinations of eigenvectors of h.

There is a difference between the cases $N = \dim G$ is odd or even. In the even case we can define a grading operator Γ which anticommutes with Q. It is given as $\Gamma = (-1)^F \psi_0^{N+1}$, where F is the fermion number operator, $\psi^a_n F + F\psi^a_n = \frac{n}{|n|}\psi^a_n$ for $n \neq 0$, and ψ_0^{N+1} is the chirality operator on the even dimensional group manifold G, with eigenvalues ± 1.

We can couple the supercharge to an external **g** valued vector potential A on the circle by setting

$$Q_A = Q + k\psi^a_n i A^a_{-n} \tag{4.11}$$

where the Fourier components of the Lie algebra valued vector potential A satisfy $(A^a_n)^* = -A^a_{-n}$. By a direct computation,

$$[S^a_n, Q_A] = kn\psi^a_n + k\lambda_{abc}\psi^b_m A^c_{n-m}. \tag{4.12}$$

This implies that for a finite gauge transformation $f \in LG$

$$S(f) Q_A S(f)^{-1} = Q_{A^f}, \tag{4.13}$$

where $A^f = f^{-1}Af + f^{-1}df$.

Theorem 2. *The family Q_A of hermitean operators in H defines an element of the twisted K-theory group $K^1(G, k')$ where the twist is $k' = (2k+1)/\theta^2$ times the generator of $H^3(G, \mathbb{Z})$.*

Proof. As pointed out in [BCMMS], twisted K-theory classes over G can be thought of as equivariant maps $f : P \to Fred_*$, where P is a principal $PU(H)$ bundle over G with a given Dixmier-Douady invariant $\omega \in H^3(G, \mathbb{Z})$. The equivariance condition is $f(pg) = g^{-1} f(p) g$ for $g \in PU(H)$. In the case at hand, the principal bundle P is obtained by embedding of the loop group $LG \subset PU(H)$ through the projective representation of LG of level $k+\frac{1}{2}$. As we saw in (4.13), the family Q_A is equivariant with respect to the (projective) loop group action. Finally, the Dixmier-Douady class determined by the level $k + 1/2$ of the projective representation is k' times the generator $\frac{1}{24\pi^2}\mathrm{tr}(g^{-1}dg)^3$ on $G = \mathcal{A}/\Omega G$.

Note that in the even case the family Q_A gives necessarily a trivial element in K^1. This follows from the existence of the operator Γ which anticommutes with the hermitean operators Q_A. Thus there is no net spectral flow for this family of operators, which is an essential feature in odd K-theory.

However, in the even case we can define elements in K^0 by the standard method familiar from the theory of ordinary Dirac operators: We can split $Q_A = Q_A^+ + Q_A^-$, using the chiral projections $\frac{1}{2}(\Gamma \pm 1)$, where $(Q_A^+)^* = Q_A^-$ is a pair of nonhermitean operators with nontrivial index theory. Either of the families $Q_A^\pm$ can be used to define an element of $K^0(G, k')$. Again, we use the observation that elements in $K^0(G, k')$ can be viewed as equivariant maps from the total space P of a principal $PU(H)$ bundle over G to the set $Fred$ of all Fredholm operators in H.

The operator Q is also of interest in cyclic cohomology. It can be used to construct the entire cyclic cocycle of Jaffe, Lesniewski, and Osterwalder [JLO] (they considered the case of abelian Wess-Zumino model). The key fact is that the operator $\exp(-sQ^2)$ is a trace class operator for any real $s > 0$; in fact, there is an explicit formula for the trace, it is equal to the product of Kac character formulas for two highest weight representations of the loop group, one in the bosonic Fock space and the second in the fermionic Fock space.

The second ingredient in cyclic cohomology is an associative algebra $\mathcal{B}$ acting in the Hilbert space such that each $[Q, a]$ is a bounded operator for $a \in \mathcal{B}$. This is the case for the elements $S(f) = f_n^a S_{-n}^a$ in the current algebra for each smooth function f on the unit circle. However, the operators $S(f)$ are not bounded. This should not be a serious problem since the norm of the restriction of $S(f)$ to a finite energy subspace is growing polynomially in energy, whereas $\mathrm{tr} e^{-sQ^2}$ is decreasing exponentially in energy. Recall that the even entire JLO cocycle is composed of terms

$$\int_{s_i > 0, \sum s_i = 1} \mathrm{tr}\, \Gamma a_0 e^{-s_0 Q^2} [Q, a_1] e^{-s_1 Q^2} \dots [Q, a_n] e^{-s_n Q^2} ds_0 \dots ds_{n-1}$$

with $a_i \in \mathcal{B}$. This is finite for elements a_i in the current algebra. The above formula can be used also in the odd case by setting $\Gamma = 1$.

Since the twisted K-theory classes above are labelled by the irreducible highest weight representations of an affine Kac-Moody algebra, it is natural to ask what is the relation of the twisted K-theory on G to the Verlinde algebra of G, on a given level k. Actually, D. Freed, M. Hopkins and C. Teleman have announced that there is a product in $K_G(G,k)$ (the G equivariant version of $K(G,k)$) which makes it isomorphic to the Verlinde algebra, [F], [FHT]. It would be interesting to understand the relation of their geometric construction to the algebraic construction based on the supersymmetric WZW model.

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