Introduction to Conformal Geometry and Quasiconformal Maps
Department of Mathematics and Statistics
Winter 2013 / Vuorinen
Exercise 10, 2013 `File: icg1310.tex, 2013-3-3,18.22`

1. Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous. The *cluster set* of f at a point $b \in \partial G$ is the set $C(f,b) = \{b' \in \overline{\mathbf{R}}^n : \exists (b_k) \in G^n, b_k \to b, f(b_k) \to b'\}$. It is clear that $C(f,b) \subset \overline{G'}$, and that for injective maps $C(f,b) \subset \partial G'$. The cluster set $C(f,b)$ is a singleton iff f has a limit at b. The cluster set is connected if there are arbitrarily small numbers $t > 0$ such that $B(b,t) \cap G$ is connected. We say that f is *boundary preserving* if $C(f,b) \subset \partial G'$ for all $b \in \partial G$.
(a) Find for each $b \in S^1$ the cluster set $C(f,b)$ of the analytic function $f : B^2 \to B^2$, with $f(z) = \exp g(z)$ when $g(z) = -(1+z)/(1-z), z \in B^2$.
(b) Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be open and continuous. Show that f is boundary preserving iff f is proper.

2. Let $f : \mathbf{B} \to f(\mathbf{B}) \subset \mathbf{R}^n$ be a homeomorphism with the property that there exists a number $K \geq 1$ such that for all $x, y \in \mathbf{B}$ $\mu_{f(\mathbf{B})}(f(x), f(y)) \leq K\mu_{\mathbf{B}}(x,y)$, and let (b_n) be a sequence of points in $\mathbf{B}$ such that $b_k \to b \in \partial\mathbf{B}$ and $f(b_k) \to \beta$. (It is known, that $\partial f\mathbf{B}$ is connected, cf. **1**.) Let $\rho(a_k, b_k) < M \,\forall k$. Show that $\lim_{k\to\infty} f(a_k) = \beta$ exists. Does the same conclusion hold for noninjective mappings?

3. Let A, B, C, D be distinct points on the unit circle S^1 in the stated order and 2α and 2β the lengths of the arcs AB and CD, respectively. Find the least value of $\mathsf{M}\big(\Delta(AB, CD)\big)$. [Hint: $|A-C||B-D| = |A-B||C-D| + |B-C||A-D|$ by Ptolemy's theorem [CG, p. 42], [BER, 10.9.2].]

4. Let $E \subset \mathbf{R}^n$ be compact, $\operatorname{cap} E > 0$ and $E(t) = \cup_{x\in E}\mathbf{B}(x,t)$. It follows from Ziemer's theorem that for a fixed $t > 0$ $\operatorname{cap}(E(t), \overline{E(s)}) \to \operatorname{cap}(E(t), E), s \to 0$. Show that $\operatorname{cap}(E(t), E) \to \infty$, when $t \to 0$. [Hint: Ziemer's theorem and 5.24[CGQM] may be helpful here.]

5. Let $f : \mathbf{B} \to \mathbf{B}$ be a homeomorphism with $f(0) = 0$ and assume that there is $K \geq 1$ such that for all distinct $x, y \in \mathbf{B}$

$$\lambda_{\mathbf{B}}(x,y)/K \leq \lambda_{f\mathbf{B}}(f(x), f(y)) \leq K\lambda_{\mathbf{B}}(x,y).$$

Prove that there are $a, b, c, d > 0$ such that $a|x|^b \leq |f(x)| \leq c|x|^d$ for all $x \in \mathbf{B}$.

6. In complex notation, Möbius transformations are defined by $T(z) = \frac{az+b}{cz+d}$ with $\Delta = ad - bc \neq 0$. These mappings generate a group.
(a) Prove that $T(z_1) - T(z_2) = \frac{\Delta(z_1-z_2)}{(cz_1+d)(cz_2+d)}$.

(b) Prove that the cross ratio $[z_1, z_2, z_3, z_4] = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}$ is invariant under T.
(c) Prove that $\frac{T''(z)}{T'(z)} = -\frac{2c}{cz+d}$, $D(\frac{T'(z)}{T''(z)}) = -\frac{1}{2}$, and $S_T = 0$,

$$S_T = \frac{T'''(z)}{T'(z)} - \frac{3}{2}\left(\frac{T''(z)}{T'(z)}\right)^2 = \left(\frac{T''(z)}{T'(z)}\right)' - \frac{1}{2}\left(\frac{T''(z)}{T'(z)}\right)^2.$$

L. V. Ahlfors writes in [A5]: “For those who like computing I recommend proving the formula:”

$$[f(z+ta), f(z+tb), f(z+tc), f(z+td)] = [a, b, c, d](1+\frac{t^2}{6}S_f(z)+O(t^3)).$$

Here f is an analytic function. This formula is not part of problem 6.

File: icg1310.tex, 2013-3-3,18.22