Introduction to Conformal Geometry and Quasiconformal Maps
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Exercise 5, 2015 `File: icg1305.tex, 2013-1-28,22.15`

1. Let $G = \mathbb{R}^n \setminus \{0\}$, $x, y \in G$, and let $\varphi \in [0, \pi]$ be the angle between the segments $[0, x]$ and $[0, y]$.
(a) Show that $\sin \frac{1}{2}\varphi \le \frac{|x-y|}{|x|+|y|}$.
(b) Show that $|x - y| \le ||x| - |y|| + 2 \min\{|x|, |y|\} \sin(\varphi/2)$.
(c) It is known (by [MOS]) that $k_G(x, y) = \sqrt{\varphi^2 + \log^2 \frac{|x|}{|y|}}$. Using this result show that there is constant A such that that $k_G(x, y) \le A\, j_G(x, y)$ for all $x, y \in G$ i.e. that G is a uniform domain.

2. Let $x, y \in \mathbb{R}^n \setminus \{0\}$ and $|y| \ge |x|$. Show that $d(y, [0, x]) \ge \frac{|x-y|}{2}$.

3. Let $f \in \mathcal{GM}(\mathbf{B}^n)$ and $r \in (0, 1)$. Show that

$$|f(x) - f(y)| \le \frac{1}{1 - r^2}|x - y|,$$

for $|x|, |y| \le r$. [Hint: $\text{sh}^2 \frac{\rho(x,y)}{2} = \ldots$]

4. For an open set D in $\mathbb{R}^n$, $D \ne \mathbb{R}^n$, let

$$\varphi_D(x, y) = \log\left(1 + \max\left\{ \frac{|x - y|}{\sqrt{d(x)d(y)}} \, , \, \frac{|x - y|^2}{d(x)d(y)} \right\}\right) \; ; \; x, \, y \in D \, .$$

Show that $j_D(x, y)/2 \le \varphi_D(x, y) \le 2\, j_D(x, y)$.

5. Let $G = \mathbb{R}^n \setminus \{0\}$ and $f(x) = a^2x/|x|^2$ for $x \in G$, where $a > 0$. Show that $k_G\big(f(x), f(y)\big) = k_G(x, y)$ and $j_G\big(f(x), f(y)\big) = j_G(x, y)$ for $x, y \in G$.

6. Let $f : G \to G' = f(G)\,, G, G' \subset \mathbb{R}^n$, be a homeomorphism such that for some $C > 0$ and all $x, y \in G$, $k_{G'}(f(x), f(y)) \le C k_G(x, y)$. Suppose that $b \in \partial G$ and that $b_k \in G$ with $b_k \to b, f(b_k) \to \beta, k \to \infty$, and let $E = \cup D(b_k, 1)$. Here $D(x, M)$ stands for the quasihyperbolic ball. Prove that $f(x) \to \beta$ when $x \to b, x \in E$. Note: By topology, for each sequence (b_k) tending to a boundary point b of G such that the image sequence also has a limit γ, it follows that $\gamma \in \partial G'$.