Introduction to Conformal Geometry and Quasiconformal Maps
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Winter 2013 / Vuorinen
Exercise 4, 2013 `File: icg1304.tex, 2013-2-11,14.01`

1. (a) Show that an inversion f in $S^{n-1}(a,r)$, when $a_n = 0$, preserves the upper half-space

$$f(\mathbb{H}^n) = \mathbb{H}^n.$$

(b) Show that the expression

$$\frac{|x-y|^2}{2x_n y_n},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, is invariant under an inversion in $S^{n-1}(a,r)$ when $a_n = 0$.

Notation for problems 2-4.
Let $L(x,y)$ be the line through the points x and y. Let $\partial\mathbb{H}$ be the real axis. The complex number x is denoted by $x = x_1 + ix_2$ or $x = |x|e^{i\alpha}$.

Let $x = x_1 + ix_2, y = y_1 + iy_2 \in \mathbb{C} \setminus \{0\}$ such that $0, x, y$ are noncollinear, and $x^* = x/|x|^2$, $y^* = y/|y|^2$. Let

$$a = i\frac{y(1+|x|^2) - x(1+|y|^2)}{2(x_2 y_1 - x_1 y_2)} \quad \text{and} \quad r_a = \frac{|x-y|\big|x|y|^2 - y\big|}{2|y||x_1 y_2 - x_2 y_1|}. \tag{1}$$

Then the four points x, y, x^*, y^* are in $S^1(a, r_a)$. Moreover, the circle $S^1(a, r_a)$ is orthogonal to S^1. These facts are assumed to be well-known.

Let $x = e^{i\alpha}, y = e^{i\beta}$ with $\alpha, \beta \in (0,\pi)$ and $\alpha \neq \beta$. The midpoint z of the hyperbolic segment $J[x,y]$ in $\mathbb{H}$ is obtained from (2.7)/CGQM

$$z = e^{i\delta}, \quad \delta = \arccos\left(\frac{\cos\frac{\beta+\alpha}{2}}{\cos\frac{\beta-\alpha}{2}}\right). \tag{2}$$

2. Let $x, y \in S^1 \cap \mathbb{H}$, $\{x_*, y_*\} = S^1 \cap \partial\mathbb{H}$, let x_*, x, y, y_* occur in this order on S^1. Let a, r_a be as in (1) and z be as in (2). Furthermore, let $w = L(x,y) \cap \partial\mathbb{H}$, $v = L(x, x_*) \cap L(y, y_*)$, $n = L(x,y) \cap S^1(\frac{w}{2}, \frac{|w|}{2}) \cap \mathbb{H}$. Then
(1) the line $L(a,z)$ is orthogonal to $\partial\mathbb{H}$.
(2) the line $L(w,z)$ is tangent to the circle S^1.
(3) the circle $S^1(a, r_a)$ is orthogonal to the circle $S^1(\frac{w}{2}, \frac{|w|}{2})$.
(4) the line $L(v,z)$ is orthogonal to $\partial\mathbb{H}$.
(5) the point v is on the circle $S^1(a, r_a)$.
(6) the point n is the midpoint of the Euclidean segment $[x,y]$.

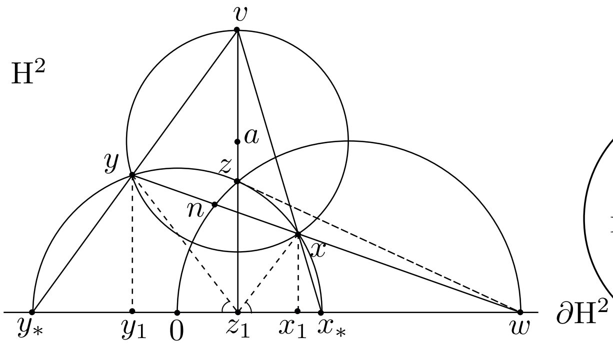

Figure 1: Problem 2

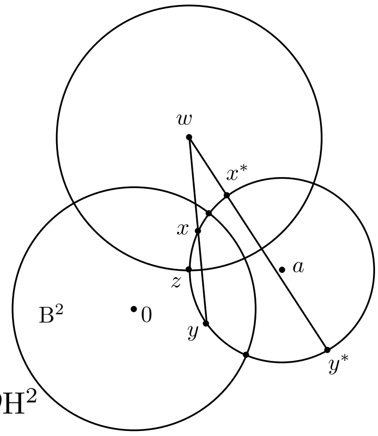

Figure 2: Problem 3

(7) $\angle y_1 z_1 y = \angle x_1 z_1 x$.

3. Let $x, y \in \mathbb{B} \setminus \{0\}$ such that $0, x, y$ are noncollinear and $|x| \neq |y|$. Let $x^* = x/|x|^2$, $y^* = y/|y|^2$ and $w = L(x,y) \cap L(x^*, y^*)$. Construct the circle $S^1(w, r_w)$, which is orthogonal to the circle $S^1(a, r_a)$, where a, r_a are as in (1). Show that the circle $S^1(w, r_w)$ is orthogonal to the circle S^1 and

$$w = \frac{y(1-|x|^2) - x(1-|y|^2)}{|y|^2 - |x|^2} \quad \text{and} \quad r_w = \frac{|x-y|\sqrt{(1-|x|^2)(1-|y|^2)}}{\big||y|^2 - |x|^2\big|}.$$

4. Fix $0 < t < 1$, let $x_* = t + i\sqrt{1-t^2}$ and $y_* = t - i\sqrt{1-t^2}$. Construct the orthogonal circle $S^1(a, r_a)$ intersecting the unit circle at the points x_*, y_*. Let $\{x, y\} = S^1(a, r_a) \cap \mathbb{B}$, $x^* = x/|x|^2$, $y^* = y/|y|^2$, and $u = L(x, y^*) \cap L(y, x^*)$. Then
(1) the point

$$u = \frac{y(1-|x|^2) + x(1-|y|^2)}{1 - |x|^2|y|^2},$$

and $\mathrm{Re}\, u = \mathrm{Re}\, w$, where $w = L(x,y) \cap L(x^*, y^*)$.
(2) the three points w, x_*, y_* are collinear and hence $\mathrm{Re}\, u = t$.

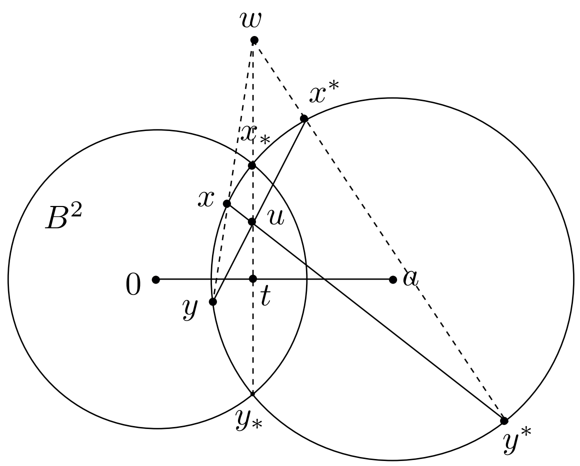

Figure 3: Problem 4