Introduction to Conformal Geometry and Quasiconformal Maps
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Exercise 3, 2013 `File: icg1303.tex, 2013-1-6,17.25`

N.B. Numbered results/ formulas refer to CGQM.

1. Show that for all $a, x, y \in B^n$

$$\frac{|T_a x - T_a y|^2}{(1-|T_a x|^2)(1-|T_a y|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}\,.$$

Hint. Lectures, Ahlfors bracket.

2. For this exercise, recall first the picture of Problem 1, Exercise 1.

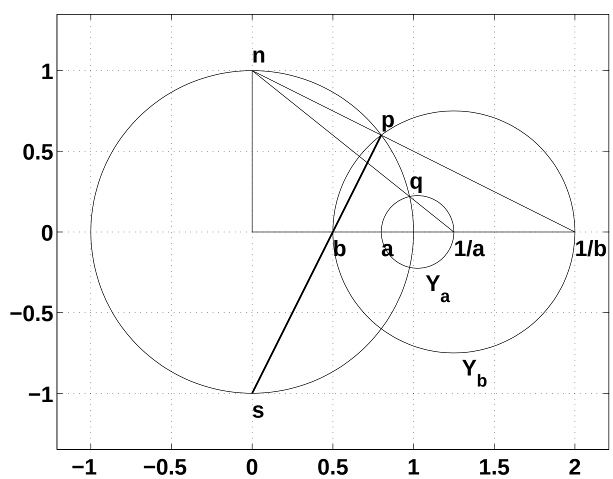

Figure 1: The unit circle and orthogonal circles Y_a, Y_b.

(a) Let $r \in (0,1)$. Find a point $a \in (0, re_1)$ such that $T_a(0) = -T_a(re_1)$.
(b) For $\varphi \in (0, \frac{1}{2}\pi)$, let $x_\varphi = (\cos\varphi, \sin\varphi)$ and $y_\varphi = (\cos\varphi, -\sin\varphi)$. Then there exists a Möbius transformation $T_a \colon \mathbb{B}^2 \to \mathbb{B}^2$ with $T_a e_1 = e_1$, $T_a(-e_1) = -e_1$, $T_a(x_\varphi) = e_2 = -T_a(y_\varphi)$. Find $|a|$.

3. (a) Let $0 < a \le 1 \le b$ and $\varphi(t) = \max\{t^a, t^b\}$. Then, with $u = \log^{1-a} 2$, $v = \log^{1-b} 2$,

(1) the function

$$f_1(t) = \frac{\log(1+t)}{\log(1+t^a)}$$

is increasing on $(0,\infty)$ with range $(0, 1/a)$.

(2) for $t \in (0,1)$

$$u \le f_2(t) < 1,$$

where

$$f_2(t) = \frac{\log(1+t^a)}{\log^a(1+t)}.$$

The function $f_2(t)$ is decreasing on $(0,1)$ and increasing on $(1, ınfty)$ with $f_2(1) = u$.

(3) the function

$$f_4(t) = \frac{\log(1+t^b)}{\log(1+t)}$$

is increasing on $(0,\infty)$ with range $(0,b)$.

4. Let $x, y \in \mathbb{R}^n$ and let t_x be a spherical isometry with $t_x(x) = 0$. Show that

$$|t_x y| = \frac{|x-y|}{\sqrt{(1+|x|^2)(1+|y|^2) - |x-y|^2}} \ .$$

Let $\alpha \in [0, \frac{1}{2}\pi]$ be such that $\sin\alpha = q(x,y)$. Then α is the angle between the segments $[e_{n+1},\, t_x x] = [e_{n+1},\, 0]$ and $[e_{n+1},\, t_x y]$ at e_{n+1} . Show that the above formula can be rewritten as $|t_x y| = \tan\alpha$.

5. For $x, y \in \mathbb{B}^n$ and $T_x \in \mathcal{M}(\mathbb{B}^n)$ show that

$$|T_x y| = \frac{|x-y|}{\sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)}} = \frac{s}{\sqrt{1+s^2}} \ ,$$

where $s^2 = |x-y|^2/((1-|x|^2)(1-|y|^2))$.

6. Let $h : [0,\infty) \to [0,\infty)$ be strictly increasing with $h(0) = 0$ such that $h(t)/t$ is decreasing. Show that $h(x+y) \le h(x) + h(y)$ for all $x > 0$. With $h(t) = t^\alpha$, $\alpha \in (0,1)$, apply this result to show that if $d(x,y)$ is a metric then $d^\alpha(x,y) = d(x,y)^\alpha$ also is a metric.