**Introduction to Conformal Geometry and Quasiconformal Maps**
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**Winter 2013 / Vuorinen**
Exercise 2, 2013 `File: icg1302.tex, 2012-12-31,16.44`
**N.B.** Numbered results/ formulas refer to CGQM.

**1.** Let $f$ be an inversion in $S^{n-1}(a,r)$ as defined in 1.2(2)[CGQM]. Show that $f^{-1}=f$ and that $|x-a||f(x)-a|=r^2$ for all $x\in\mathbb{R}^n\setminus\{a\}$. By considering similar triangles show that the following identity holds for $x,y\in\mathbb{R}^n\setminus\{a\}$ :

$$|f(x)-f(y)|=\frac{r^2|x-y|}{|x-a||y-a|}\,.$$

**2.** (a) For $0<t<1$ let $w(t)=t/\sqrt{1-t^2}$. Show that $q(0,w(t)e_1)=t$ and that

$$\frac{t}{s}<\frac{w(t)}{w(s)}<\frac{2t}{s}$$

for $0<s<t<\frac{1}{2}\sqrt{3}$.
(b) Let $q(A)=\sup\{q(x,y)\colon x,y\in A\}$ for $A\subset\overline{\mathbb{R}^n}$. Show that

$$q(Q(z,r))=q(\partial Q(z,r))=2r\sqrt{1-r^2}$$

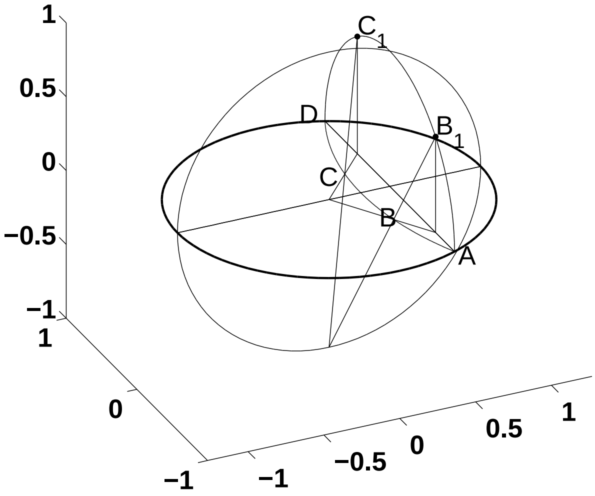
for $0<r\le 1/\sqrt{2}$.

**3.** (a) Show that if $a,x\in\mathbb{B}^n$, then $\sigma_a(x)\in\mathbb{B}^n$ [CGQM, (1.36)].
(b) Let $h(w)=r^2w/|w|^2, r>0, w\in\mathbb{R}^n\setminus\{0\}$. Show that if $x,y\in\mathbb{R}^n\setminus\{0\}, |x|\le|y|, \lambda=(|x|+|x-y|)/|x|, z=\lambda x$, then

$$|h(x)-h(z)|\le|h(x)-h(y)|\le 3|h(x)-h(z)|\,.$$

**4.** Show that the inversion in $S^{n-1}(-e_n,\sqrt{2})$ maps $W=\{z\in\mathbb{R}^n:|z|<1\ \ \&\ \ z_n=0\}$ onto $S^{n-1}\cap\mathbb{H}^n$, where $\mathbb{H}^n=\{z\in\mathbb{R}^n: z_n>0\}$. For $B,C\in W\setminus\{0\}$ consider the circular arc $U\subset W$ with $B,C\in U$ and perpendicular to $\partial B^n$ at the points $A$ and $D$ ($A,B,C,D$ are on $U$ in this order). Let $B_1,C_1\in S^{n-1}$ be the image points of $B,C$ under the inversion. Let $B_2$ be the projection of $B_1$ onto the segment $[A,D]$. By looking at the picture, show that

- $0,B_1,B_2$ are collinear.
- $|A,B,C,D|=|A,B_1,C_1,D|\,.$




**5.** Let $h(x) = x/|x|^2$. Show that $h$ maps the sphere $S^{n-1}(be_1, s)$ (assume that $b > 1 + s$) onto a sphere. Hint. Write $u = (b - s)e_1, v = (b + s)e_1$. If the image is a sphere $S^{n-1}(c, t)$, then clearly $c = (h(u) + h(v))/2$ and $t = |h(u) - h(v)|/2$. Hence it remains to show that $|z - be_1| = s$ implies $|h(z) - c| = t$.

**6.** The lines $[-e_1, 0]$ and $[ae_1, \infty]$, $a > 0$, can be mapped onto $[-e_1, e_1]$ and $[be_1, \infty] \cup [-be_1, \infty]$ by a Möbius transformation. Give a definition for $b$ in terms of $a$. Notice that $[x, \infty] = \{xt : t \geq 1\}$, if $x \in \mathbb{R}^n \setminus \{0\}$.