Introduction to Conformal Geometry and Quasiconformal Maps
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Exercise 1, 2013 `File: icg1301.tex, 2013-1-6,17.28`
N.B. Numbered results/ formulas refer to CGQM.

1. Let $0 < a < 1, b = a/(1+\sqrt{1-a^2})$ and for $0 < t < 1$ let Y_t be the circle centered at $(t+1/t)/2$ and the segment $[t, 1/t]$ as its diameter. Show Y_t is orthogonal to the unit circle and that Y_b intersects the unit circle at the point $p = (a, \sqrt{1-a^2})$. Show that the point $(b, 0)$ is on the segment $[p, s]$, $s = (0, -1)$. Find the point $q = (q_1, \sqrt{1-q_1^2})$ in the upper half plane where Y_a intersects the unit circle and show that $(0, 1), q, (1/a, 0)$ are collinear.

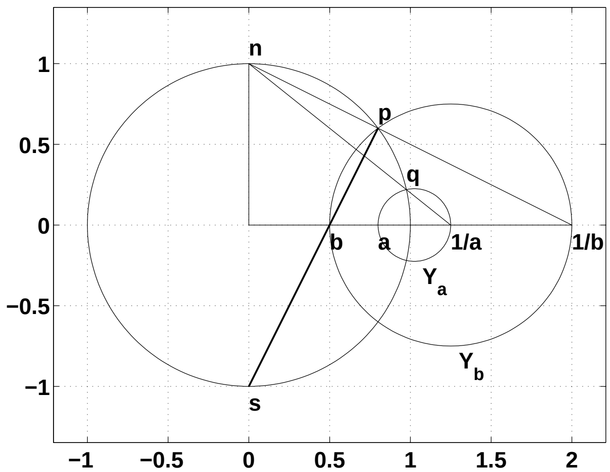

Figure 1: The unit circle and orthogonal circles Y_a, Y_b.

2. Let $G \subset \mathbb{R}^n$ be a domain and $f : G \to \mathbb{R}$ a continuous function. Fix $b \in \partial G$. We say that f has a sequential limit α at b, if there exists a sequence (b_k) in G with $b_k \to b$ and $f(b_k) \to \alpha$ when $k \to \infty$. Let $C(f, b)$ be the set of all sequential limits of f at b. Then obviously $C(f, b) \subset \overline{f(G)}$.

(a) Show that f has limit $\overline{f}(b)$ at b iff $C(f, b) = \{\overline{f}(b)\}$.

(b) Show that $C(f, b) = \cap_{k=1}^{\infty} \overline{f(G \cap B^n(b, 1/k))}$, if $b \neq \infty$.

(c) Let $G = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $f : G \to (0, \infty)$ be the function $\overline{\text{arc}} \tan(y/x) \in (0, \pi)$. What is $C(f, 0)$?

(d) Consider the map $f : B^2 \to B^2, f(z) = \exp((z+1)/(z-1))$. What is $C(f, 1)$?

3. Find a Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

which maps $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \colon y > 0\}$ onto $B^2 = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$ such that $(-1, 0, 1) \mapsto (1, i, -1)$ $[i = (0, 1)]$.

4. A mapping $f \colon \mathbb{R} \to \mathbb{R}$ is said to be Hölder continuous, if there exist constants $C, \beta > 0$ such that $|f(x) - f(y)| \leq C|x - y|^\beta$ for all $x, y \in \mathbb{R}$. Let $f \colon \mathbb{R} \to \mathbb{R}$ be Hölder continuous with exponent $\beta > 1$. Show that f is a constant, equal to $f(0)$.

5.

(a) Let $G, G' \subset \mathbb{R}^n$ be domains and $f : G \to G'$ be a homeomorphism. For $x, y \in G$ define $m_f(x, y) = |f(x) - f(y)|$. Show that m_f is a metric.

(b) Let $\rho \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $\mathbb{R}_+ = (0, \infty)$, be defined by $\rho(x, y) = |\log(x/y)|$. Show that ρ is a metric.

(c) Suppose that $u \colon (\mathbb{R}_+, \rho) \to (\mathbb{R}_+, \rho)$ is uniformly continuous. Show that $u(x) \leq Ax^B$ for some constants A and B for all $x \geq 1$.

6. Show the following facts.

(a) Distinct points $z_1, z_2, z_3 \in \mathbb{C}$ are collinear iff $(z_1 - z_3)/(z_2 - z_3) \in \mathbb{R}$.

(b) Given three points $z_1, z_2, z_3 \in \mathbb{C}$, find a formula for the area of the positively oriented triangle $[z_1, z_2, z_3]$. Hint: One version of the formula is

$$|z_1(\overline{z_2} - \overline{z_3}) + z_2(\overline{z_3} - \overline{z_1}) + z_3(\overline{z_1} - \overline{z_2})|/4.$$

Another variant is:

$$\frac{1}{2}\mathrm{Im}(\overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1)\,.$$

(c) Given three noncollinear points $z_1, z_2, z_3 \in \mathbb{C}$, find a formula for the center z of the circle through these points. Hint. There are various ways to write this. For instance, $z = D/d$ where

$$D = |z_1|^2(z_2 - z_3) + |z_2|^2(z_3 - z_1) + |z_3|^2(z_1 - z_2)\,,$$

$$d = z_1(\overline{z_3} - \overline{z_2}) + z_2(\overline{z_1} - \overline{z_3}) + z_3(\overline{z_2} - \overline{z_1})\,.$$