Elements of
DESCRIPTIVE SET THEORY

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In these notes we go through the very basic results from descriptive set theory. In our approach to the topic, the use of games is more explicit than what is usual in the literature. We do not assume any previous knowledge on set theory nor on first-order logic. For the history of the topic, see [Mo] (Borel* sets are due to D. Blackwell and Vaught codes are due to R. Vaught). For further reading [Ke] is recommended.

Contents

1. Preliminaries from set theory

Our approach to set theory is naïve i.e. we do not introduce any formal language. So when we talk on a general level about properties of sets, usually denoted by ϕ, ψ etc., we do not specify what these are. What we really mean by these, are properties that are expressible by a first-order formula with parameters. But since (at least) practically all mathematical properties are such, we do not bother to be more precise here. (Here we are dangerously close to the philosophy of mathematics and we do not want to elaborate further.) Also our notation seems to suggest that all classes i.e. the totalities of all sets that have some given property ϕ, are mathematical objects. But not all of them are, e.g. V, the class of all sets (one can choose $x = x$ as the defining property), is not, at least not from the point of view of set theory (again we do not want to elaborate further). However, a closer study shows that the way we use V and other proper classes (i.e. classes that are not sets) is done only to simplify our notations. Those readers that are not comfortable with our approach are invited to consult [Je].

1.1 Axioms

The membership relation within the elements of V is denoted by $\in$ (so if a is a set and $b \in a$, also b is a set). In addition, for classes C and sets a we write $a \in C$ meaning that a has the defining property of C.

I Extensionality: If sets a and b have the same elements, then $a = b$.

Notice, that also the inverse of the implication in Extensionality holds. And that from now on to determine a set, it is enough to describe its elements, e.g. $\{3, 8, i\}$, $\{n \in \mathbb{N} |\ n$ is even$\}$,..., and of course $\emptyset$. Also we extend the idea in Extensionality to classes i.e. two classes are considered the same if they have the same elements and a class and a set are considered the same if they have the same elements.

1.1.1 Exercise. *Show that every set is a class.*

II Foundation: Every non-empty set a has an $\in$-minimal element i.e. there is $x \in a$ such that for all $y \in a$, $y \notin x$.

III Pairing: For any sets a and b, $\{a, b\}$ is a set.

Notice, that from Pairing it follows that for every set a, $\{a\}$ is a set.

1.1.2 Exercise. *Show that there is no set a such that $a \in a$ or sets a and b such that $a \in b \in a$.*

IV Separation: If a is a set and ϕ is a property, then $\{x \in a |\ \phi(x)\}$ is a set, where $\phi(x)$ means that x has the property ϕ.

V Union: For every set a, the union $\cup a$ of the elements of a is a set ($x \in \cup a$ if $x \in b$ for some $b \in a$).

1.1.3 Exercise.
(i) Show that if a, b, c, d *and* e *are sets, then* $\{a, b, c, d, e\}$ *is a set.*
(ii) We write (a, b) *for the set* $\{a, \{a, b\}\}$. *Show that*
(a) (a, b) *is indeed a set,*
(b) if $(a, b) = (c, d)$, *then* $a = c$ *and* $b = d$.

VI Power Set: For every set a, the power set $P(a)$ of a is a set ($x \in P(a)$ if $x \subseteq a$).

So far we have had no axiom that states that there exists even a single set. The next axiom says that there is an infinite set. However, it seems to assume that the empty set already exists. So should we not have an axiom that says this? We could have such an axiom but usually it is not included. The axiomatic version of set theory is developed within the first-order logic and there it is common to define the proof system so that one can always prove that there exists x such that $x = x$ (in basic logic courses it is explained why it is done this way). So in the case of set theory, one can always prove the existence of at least one set.

1.1.4 Exercise.
(i) Show that the empty set $\emptyset$ *exists.*
(ii) For sets a *and* b, *show that* $a \times b = \{(x, y) |\ x \in a,\ y \in b\}$ *is a set.*

VII Infinity: There exists an inductive set i.e. a set a such that $\emptyset \in a$ and if $x \in a$, then also $x \cup \{x\} \in a$ (exercise: show that $x \cup \{x\}$ is a set).

When we talk about functions f from a set a to a set b, we always mean that $f = \{(x, f(x)) |\ x \in a\}$ is a set. We talk also about class functions:

1.1.5 Definition. *Let* C *be a class. We say that a function* $F : C \to V$ *is a class function if the graph of* F *is a class i.e. there is a property* ϕ *such that for all sets* x, x *has the property* ϕ *iff* $x = (a, F(a))$ *for some set* $a \in C$.

VIII Replacement: If a is a set and $F : a \to V$ is a class function, then $\{F(x) |\ x \in a\}$ is a set.

1.1.6 Exercise.
(i) Show that if a *is a set and* $F : a \to V$ *is a class function, then* F *is a set.*
(ii) Show that if a *and* b *are sets and* $f : a \to b$ *is a function, then it is a class function.*

IX Choice: If a is a set and every $x \in a$ is non-empty, then there is a function $f : a \to \cup a$ such that for all $x \in a$, $f(x) \in x$.

The theory that consists of all these axioms is called ZFC. If the Choice is left out, the resulting theory is called just ZF. Unless we state otherwise, we work in ZFC.

1.2 Recursive definitions

1.2.1 Definition.

(i) If C *is a class, then a class* $<$ *is called a partial ordering of* C *if the elements of* $<$ *are of the form* (x,y), $x,y\in C$, *and the following holds: if* $(x,y),(y,z)\in<$, *then* $(x,z)\in<$ *and* $(y,x)\notin<$. *Instead of writing* $(x,y)\in<$, *we will simply write* $x<y$.

(ii) A partial ordering $<$ *is a linear ordering, if in addition, for all* $x,y\in C$, $x<y$ or $x=y$ or $y<x$.

(iii) A partial ordering is well-founded if for all $x\in C$, $\{y\in C|\ y<x\}$ *is a set and if* a *is a non-empty set such that every element of it belongs to* C, *then* a *has a* $<$*-minimal element. If in addition the partial ordering is a linear ordering, it is called a well-ordering.*

1.2.2 Theorem. *Suppose* C *is a class and* $<$ *is a well-founded partial ordering of* C. *Let* ϕ *be a property and assume that for all* $x\in C$, *if every element of* $\{y\in C|\ y<x\}$ *has the property* ϕ, *then also* x *has it. Then every element of* C *has the property* ϕ.

Proof. Suppose not. Let $x\in C$ be such. We show first that we can choose x so that it is $<$-minimal element of C among those that do not have the property ϕ: If x is not such then the class a of all element of C which are smaller than x and do not have the property ϕ is non-empty and a set. Since $<$ is well founded, a has a $<$-minimal element. Clearly this is as wanted.

But if x is a minimal among those that do not have the property ϕ, then every element of $\{y\in C|\ y<x\}$ has the property, and so also x has it, a contradiction. $\square$

1.2.3 Theorem. *Suppose* C *is a class,* $<$ *is a well-founded partial ordering of* C *and* $G:V\to V$ *is a class function. Then there is a unique class function* $F:C\to V$ *such that for all* $x\in C$, $F(x)=G(F\restriction C_x)$, *where* $C_x=\{y\in C|\ y<x\}$.

Proof. We say that $A\subseteq C$ is downward closed if $x<y\in A$ implies $x\in A$. We start with an exercise:

1.2.3.1 Exercise. *Suppose that a set* $A\subseteq C$ *is downward closed and* $f,g:A\to V$ *(recall Exercise 1.1.6) are such that for all* $z\in A$, $f(z)=G(f\restriction C_z)$ *and* $g(z)=G(g\restriction C_z)$ *(notice that* $C_z\subseteq A$*). Show that* $f=g$. *Conclude that if* F *exists, it is unique.*

Now let ϕ the following property of sets a: a is of the form (x,y) where $x\in C$ and y is such that there is a function $f_x:C_x\to V$ such that $y=G(f_x)$ and for all $z\in C_x$, $f(z)=G(f_x\restriction C_z)$. We will show that for every $x\in C$, there is a set y such that (x,y) has the property ϕ. Then since by Exercise 1.2.3.1, such y is unique, ϕ defines a class function $C\to V$.

To see that y exists, it is enough to show that f_x exists. We prove this by induction i.e. by using Theorem 1.2.2. So suppose that the claim holds for every $z\in C_x$. We notice

(*) if $z,w\in C$ and f_z and f_w exist, then $f_z\restriction(C_z\cap C_w)$ and $f_w\restriction(C_z\cap C_w)$ satisfy the requirements of Exercise 1.2.3.1 for $A=(C_z\cap C_w)$ and thus $f_z\restriction(C_z\cap C_w)=f_w\restriction(C_z\cap C_w)$.

So by (*), if C_x does not have maximal elements ($z \in C_x$ is maximal if there are no $y \in C_x$ such that $z < y$) $f_x = \bigcup_{z<x} f_z$ is as wanted. On the other hand, if C_x has maximal elements, we let $f_x = \cup\{f_z \cup \{(z, G(f_z))\} |\ z \in C_x \textit{ is maximal}\}$. Again by (*), f_x is as wanted.

So we are left to prove that for all x, $F(x) = G(F \restriction C_x)$. So suppose that this holds for all $z \in C_x$ and let f_x be as in the definition of ϕ. Then by Exercise 1.2.3.1, $F \restriction C_x = f_x$ and thus $F(x) = G(f_x) = G(F \restriction C_x)$. ▫

1.3 Ordinals

1.3.1 Definition.

(i) We say that a set a is transitive if $x \in y \in a$ implies $x \in a$ (i.e. $\cup a \subseteq a$).

(ii) We say that a set α is an ordinal if it is transitive and linearly ordered by $\in$. For ordinals α and β, one usually writes $\alpha < \beta$ instead of $\alpha \in \beta$ and $\alpha \leq \beta$ for $\alpha < \beta$ or $\alpha = \beta$.

(iii) The class of all ordinals is denoted by On.

1.3.2 Exercise.

(i) Show that ordinals are well-ordered by $\in$.

(ii) Show that $0 = \emptyset$ is an ordinal.

(iii) Show that if α is an ordinal, then also $\alpha + 1 = \alpha \cup \{\alpha\}$ is an ordinal.

(iv) Show that if a is a set of ordinals and for all $\alpha, \beta \in a$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$, then $\cup a$ is an ordinal.

(v) Show that if α is an ordinal and $\beta \in \alpha$, then β is an ordinal.

(vi) Show that if α and β are ordinals, then so is $\alpha \cap \beta$.

1.3.3 Lemma. *Let α and β be ordinals.*

(i) If $\alpha \subseteq \beta$, then either $\alpha = \beta$ or $\alpha \in \beta$.

(ii) Either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Proof. (i): Suppose $\alpha \neq \beta$. Then $\beta - \alpha$ is not empty and thus it has the least element γ. If $\delta \in \gamma$, then $\delta \in \beta$ and so by the choice of γ, $\delta \in \alpha$. On the other hand, if $\delta \in \alpha$, then $\gamma \not\leq \delta$, because otherwise $\gamma \in \alpha$ and this is against our choice of γ. Thus since $\in$ linearly orders β, $\delta \in \gamma$. It follows that $\alpha = \gamma$ and so $\alpha \in \beta$.

(ii): Now by Exercise 1.3.2 (vi), $\gamma = \alpha \cap \beta$ is an ordinal. Then $\gamma = \alpha$ or $\gamma = \beta$ because otherwise by (i), $\gamma \in \alpha \cap \beta = \gamma$. In the first case $\alpha \subseteq \beta$ and in the other case $\beta \subseteq \alpha$. ▫

1.3.4 Exercise.

(i) Show that On is well-ordered by $\in$.

(ii) Show that $\alpha + 1$ is the least ordinal strictly greater than the ordinal α.

(iii) For a set a of ordinals show that $\cup a$ is the supremum of a (in particular, $\cup a$ is an ordinal).

1.3.5 Definition.

(i) We say that an ordinal α is a successor ordinal if $\alpha = \beta + 1$ for some ordinal β and otherwise α is called a limit ordinal. However, 0 is usually not considered a limit ordinal.

(ii) By ω we denote the least limit ordinal $\neq 0$ (if such ordinal exists).

1.3.6 Lemma. *For every ordinal β there is a limit ordinal $\alpha > \beta$.*

Proof. We show first that ω exists. By Infinity, there is an inductive set b. Let $a = b \cap On$ and $\alpha = \cup a$. By Exercise 1.3.4 (iii), α is an ordinal. Also it is easy to see that a is inductive and thus α can not be a successor ordinal. So in particular ω exists.

Now for given ordinal β, choose a function $f : \omega \to On$ so that $f(0) = \beta$ and for successor ordinals $\gamma + 1 \in \omega$, $f(\gamma + 1) = f(\gamma) + 1$ (exercise: show that f exists and $rng(f) \subseteq On$, keep in mind that every ordinal in ω excluding 0, is a successor ordinal). Let $\alpha = \cup rng(f)$. Clearly α is as wanted. □

1.3.7 Exercise. *Show that there is no class function $f : \omega \to V$ such that for all $n \in \omega$, $f(n+1) \in f(n)$.*

1.3.8 Theorem. *For every set a there is an ordinal α and a one-to-one and onto function $f : \alpha \to a$.*

Proof. Let b be the set of all non-empty subsets of a and g be the choice function for b. We define a class function $G : On \to V$ so that for all ordinals β and functions $h : \beta \to a$ with $rng(h) \neq a$, $G(h) = g(a - rng(h))$ and for all other sets x, $G(x) = a$. Let $F : On \to V$ be such that for all ordinals γ, $F(\gamma) = G(F \restriction \gamma)$ (by Theorem 1.2.3) and suppose that for some ordinal γ, $F(\gamma) = a$. Then by letting α be the least such ordinal, α and $f = F \restriction \alpha$ are clearly as wanted.

So it is enough to show that for some γ, $F(\gamma) = a$. Suppose not. Then (by Separation) F^{-1} is a class function from a subset of a onto On. Thus by Replacement On is a set. Thus $\beta = \cup On$ is an ordinal. So $\beta \in \beta + 1 \in On$ and thus $\beta \in \beta$, a contradiction. □

1.3.9 Exercise. *(Zermelo's well-ordering theorem) Every set can be well-ordered.*

In fact, under e.g. ZF, Zermelo's well-ordering theorem is equivalent with Choice: To get Choice, simply choose a well-ordering $<$ for $\cup a$ and then for every $x \in a$, let $f(x)$ be the $<$-least element of x.

The sets V_α in the next exercise form so called cumulative hierarchy.

1.3.10 Exercise. *We define V_α for all ordinals α as follows: $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ and for limit ordinals α, $V_\alpha = \cup_{\gamma<\alpha} V_\gamma$. Show that*

(i) $\alpha \mapsto V_\alpha$ is a class function,

(ii) for $\gamma < \alpha$, $V_\gamma \subseteq V_\alpha$,

(iii) for all sets a there is an ordinal α such that $a \in V_\alpha$.

1.4 Cardinals

1.4.1 Definition. *We say that sets a and b have the same cardinality, if there is a one-to-one and onto function $f : a \to b$.*

1.4.2 Exercise.

(i) Show that the equicardinality relation from Definition 1.4.1 is an equivalence relation.

(ii) Show that if there is an onto function $f : a \to b$, then there is a one-to-one function $g : b \to a$ and vice versa assuming that $b \neq \emptyset$.

1.4.3 Theorem. *(Cantor-Bernstein) For all sets a and b, if there are one-to-one functions $f : a \to b$ and $g : b \to a$, then a and b have the same cardinality.*

Proof. For all $n \in \omega$, we define sets A_n and B_n as follows: $A_0 = a$, $B_0 = b$, $A_{n+1} = g(f(A_n))$ and $B_{n+1} = f(g(B_n))$. Finally, let $A = \bigcap_{n<\omega} A_n$ and $B = \bigcap_{n<\omega} B_n$. Clearly, for $n < \omega$, $A_{n+1} \subseteq A_n$ and $B_{n+1} \subseteq B_n$. Also (e.g. draw a picture) $f \restriction (A_n - g(B_n))$ is one-to-one function from $A_n - g(B_n)$ onto $f(A_n) - B_{n+1}$, $g^{-1} \restriction (g(B_n) - A_{n+1})$ is one-to-one function from $g(B_n) - A_{n+1}$ onto $B_n - f(A_n)$ and $f \restriction A$ is one-to-one function from A onto B. By putting these together, the required one-to-one and onto function is found. □

1.4.4 Definition.

(i) We say that an ordinal α is a cardinal if there are no $\beta < \alpha$ and a one-to-one function from α to β.

(ii) We say that a set a is finite, if for all one-to-one functions $f : a \to a$, $rng(f) = a$.

1.4.5 Lemma. *ω and every $n \in \omega$ are cardinals. In fact, every $n \in \omega$ is finite.*

Proof. We start by proving the claim for the elements of ω. Clearly it is enough to show that they are finite. We prove this by induction (i.e. using Theorem 1.2.2, keeping in mind that all elements of ω, excluding 0, are successor ordinals and, in fact, the claim we prove is that every ordinal α is either finite or $\geq \omega$).

For $n = 0$, this is clear. So suppose that this holds for n and let $f : n+1 \to n+1$ be one-to-one. For a contradiction suppose that $rng(f) \neq n+1$. By applying a transposition, we may assume that $n \notin rng(f)$. But then $f \restriction n$ is a one-to-one function from n to a proper subset of n, a contradiction.

If ω is not a cardinal, then there are $n \in \omega$ and a one-to-one function $f : \omega \to n$. But then $f \restriction n+1$ contradicts what we just proved. □

1.4.6 Exercise.

(i) Show that an ordinal α is finite iff $\alpha \in \omega$.

(ii) Show that all infinite cardinals are limit ordinals.

(iii) Show that if a is a set of cardinals, then $\cup a$ is a cardinal.

1.4.7 Lemma. *For every set a, there is a unique cardinal κ for which there is a one-to-one function from κ onto a.*

Proof. Clearly there cannot be more than one such cardinal. So we prove just the existence: Let κ be the least ordinal such that there is a one-to-one function f from κ onto a (such κ exists by Theorem 1.3.8). It is enough to show that κ is a cardinal. If not, then there is $\alpha < \kappa$ and a one-to-one function $g : \kappa \to \alpha$. By Cantor-Bernstein, we can choose g so that it is also onto. But then α and $f \circ g^{-1}$ witness that κ was not minimal. □

1.4.8 Definition. *Let a be a set. The unique cardinal κ for which there is a one-to-one function from κ onto a, is called the cardinality of a and is denoted by $|a|$. If the cardinality of a set is $\leq \omega$, we say that the set is countable.*

1.4.9 Exercise.

(i) Show that a set a *is finite iff* $|a| \in \omega$ *.*

(ii) Show that $|a| \leq |b|$ *iff there is a one-to-one function* $f : a \to b$ *.*

The elements of ω are called natural numbers and thus ω is called also the set of natural numbers i.e. $\mathbb{N}$. We also write $0 = \emptyset$ as already mentioned and $1 = 0 + 1 = 0 \cup \{0\}$, $2 = 1 + 1$, $3 = 2 + 1$ etc. Recall that for all $n \in \omega$, $n = \{0, 1, ..., n - 1\}$.

For the rest of this section, we will concentrate on the cardinal ω. From the appendix one can find these basic facts proved also for uncountable cardinals.

1.4.10 Lemma. $|\omega \times \omega| = \omega$.

Proof. Clearly $|\omega \times \omega| \geq \omega$ and thus it is enough to find a one-to-one function $f : \omega \times \omega \to \omega$. E.g. $f(n, m) = 2^n 3^m$ is such function. □

As a hint for the item (i) in next exercise we want to mention that the claim in the item can not be proved without Choice. If Choice is not assumed, it is possible that the set of reals is a countable union of countable sets and we will see later that the set of reals is not countable and this can be proved without Choice.

Also, instead of talking about functions $f : I \to X$ for some sets I and X, it is sometimes notationally convenient to talk about indexed sequences $(x_i)_{i \in I}$. So by an indexed sequence $(x_i)_{i \in I}$ we simply mean a function $f : I \to V$ such that for all $i \in I$, $f(i) = x_i$. Thus for $x : a \to V$, we sometimes also write x_i in place of $x(i)$.

1.4.11 Exercise.

(i) Suppose that a *is a countable set such that also every element of it is countable. Show that* $\cup a$ *is countable.*

(ii) Show that there are sets $X_i \subseteq \omega$, $i \in \omega$, *such that for all* i, X_i *is infinite, for all* $i \neq j$, $X_i \cap X_j = \emptyset$ *and* $\bigcup_{i<\omega} X_i = \omega$.

(iii) Show that the set of rational numbers is countable.

For sets a and b, by a^b we mean the set of all functions from b to a (e.g. $\mathbb{N}^n$). If $b = \beta$ is an ordinal we also write $a^{<\beta}$ for $\bigcup_{\alpha<\beta} a^\alpha$ and $a^{\leq\beta}$ for $\bigcup_{\alpha\leq\beta} a^\alpha$. On the level of notation, we also identify $f : 2 \to X$ with $(f(0), f(1))$ and thus think that $X \times X$ is the same as X^2, see the discussion on indexed sequences above.

1.4.12 Lemma. $|P(\omega)| = |2^\omega| = |2^{\omega\times\omega}| = |(2^\omega)^\omega| = |\omega^\omega|$.

Proof. For $|P(\omega)| = |2^\omega|$, just map every $a \subseteq \omega$ to its characteristic function. $|2^\omega| = |2^{\omega\times\omega}|$ is clear by Lemma 1.4.10. To find a one-to-one function F from $2^{\omega\times\omega}$ onto $(2^\omega)^\omega$, simply for $\eta \in 2^{\omega\times\omega}$ let $\xi = F(\eta)$ be such that for all $n, m < \omega$, $(\xi(n))(m) = \eta(n, m)$. Since $2^\omega \subseteq \omega^\omega$, $|2^\omega| \leq |\omega^\omega|$. Finally since $|\omega| \leq |2^\omega|$, it is easy to see that $|\omega^\omega| \leq |(2^\omega)^\omega|$. □

One often denotes $|2^\omega|$ by just 2^ω. It is clear from the context which possibility we mean.

1.4.13 Theorem. $|P(\omega)| > \omega$.

Proof. By Lemma 1.4.12, it is enough to show that $2^\omega > \omega$. For a contradiction, suppose $2^\omega \leq \omega$. Clearly, $2^\omega \geq \omega$ and thus, under the counter assumption, there is a one-to-one function f from ω onto 2^ω. Denote $f(n)$ by ξ_n (i.e. we enumerate 2^ω).

Let $g : \omega \to 2$ be such that for all $n < \omega$, $g(n) = 1 - \xi_n(n)$. Then $g \in 2^\omega$ and so for some $m < \omega$, $g = \xi_m$. Now $g(m) = 1 - \xi_m(m) = 1 - g(m)$, a contradiction. □

So in particular, there is a cardinal strictly greater than ω. The least such is denoted by ω_1.

1.4.14 Definition. *Let κ be an infinite cardinal.*

(i) The cofinality $cf(\kappa)$ of κ is the least ordinal α such that there is a function $f : \alpha \to \kappa$ such that $\cup rng(f) = \kappa$.

(ii) κ is called regular if $cf(\kappa) = \kappa$.

1.4.15 Exercise.

(i) Show that for all infinite cardinals κ, $cf(\kappa)$ is a regular cardinal.

(ii) Show that ω and ω_1 are regular.

1.4.16 Fact.

(i) ω is the only regular limit cardinal whose existence is provable in ZFC (see Definition A.5 in the appendix).

(ii) ZFC does not prove or disprove CH (continuum hypothesis) i.e. the claim that $2^\omega = \omega_1$ (assuming ZFC is consistent).

1.4.17 Exercise.

(i) Define $f : \omega_1 + 1 \to On$ so that $f(0) = 0$, $f(\alpha + 1)$ is the least limit ordinal $> f(\alpha)$ and for limit α, $f(\alpha) = \bigcup_{\gamma<\alpha} f(\gamma)$. Show that $f(\omega_1) = \omega_1$.

(ii) Suppose that $|a| = \omega_1$ and every $b \in a$ is countable. Show that $|\cup a| \leq \omega_1$. Hint: Use (i).

(iii) Show that if a is countable and $<$ is a well-ordering of a, then there is $f : a \to \omega_1$ such that for all $x, y \in a$, if $x < y$, then $f(x) < f(y)$.

1.5 Recursive definitions revisited

1.5.1 Definition. *Suppose X is a set.*

(i) Suppose α is an ordinal, $f : X^\alpha \to X$ is a function and $C \subseteq X$. We say that C is closed under f if for all $x \in C^\alpha$, $f(x) \in C$.

(ii) Suppose $Y \subseteq X$ and for all $i \in I$, α_i is an ordinal and $f_i : X^{\alpha_i} \to X$ is a function. Then by $C(Y, f_i)_{i\in I}$ we mean the $\subseteq$-least subset C of X such that it contains Y and is closed under every f_i, $i \in I$ (if such C exists).

1.5.2 Lemma. *Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii). Then $C(Y, f_i)_{i\in I}$ exists.*

Proof. Just let $C(Y, f_i)_{i\in I}$ be the intersection of all sets $C \subseteq X$ which contain Y and are closed under every f_i (notice that X is such a set). □

1.5.3 Lemma. *Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii). Suppose that ϕ is a property, every element of Y has it and for all $k \in I$ and $x \in C(Y, f_i)^{\alpha_k}_{i\in I}$ the following holds: If every x_j, $j < \alpha_k$, has the property, then also $f_k(x)$ has the property. Then every element of $C(Y, f_i)_{i\in I}$ has the property ϕ.*

Proof. Let C be the set of all elements of $C(Y, f_i)_{i\in I}$ that have the property ϕ. Then C contains Y and is closed under every f_i. Thus $C(Y, f_i)_{i\in I} \subseteq C$. □

1.5.4 Definition. *Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii). For all ordinals α, we define $C_\alpha(Y, f_i)_{i\in I}$ as follows:*
(i) $C_0(Y, f_i)_{i\in I} = Y$,
(ii) $C_{\alpha+1}(Y, f_i)_{i\in I} = C_\alpha(Y, f_i)_{i\in I} \cup \{f_i(x)|\ i \in I,\ x \in (C_\alpha(Y, f_i)_{i\in I})^{\alpha_i}\}$,
(iii) if α is limit, then $C_\alpha(Y, f_i)_{i\in I} = \bigcup_{\beta<\alpha} C_\beta(Y, f_i)_{i\in I}$.

1.5.5 Exercise. *Show that $\alpha \mapsto C_\alpha(Y, f_i)_{i\in I}$ is a class function from On to $P(X)$ and that for all ordinals $\alpha < \beta$, $Y \subseteq C_\alpha(Y, f_i)_{i\in I} \subseteq C_\beta(Y, f_i)_{i\in I} \subseteq C(Y, f_i)_{i\in I}$.*

1.5.6 Lemma. *Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii). Suppose further for all $i \in I$, $\alpha_i < \omega_1$. Then $C(Y, f_i)_{i\in I} = C_{\omega_1}(Y, f_i)_{i\in I}$.*

Proof. By Exercise 1.5.5, it is enough to show that $C_{\omega_1}(Y, f_i)_{i\in I}$ is closed under every f_k, $k \in I$. For this let $x \in (C_{\omega_1}(Y, f_i)_{i\in I})^{\alpha_k}$. Since ω_1 is regular, there is $\gamma < \omega_1$ such that $x \in (C_\gamma(Y, f_i)_{i\in I})^{\alpha_k}$ (Exercise, think of function $g : \alpha_k \to \omega_1$ such that for all $\beta < \alpha_k$, $g(\beta)$ is the least ordinal δ for which $x_\beta \in C_\delta(Y, f_i)_{i\in I}$). But then $f_i(x) \in C_{\gamma+1}(Y, f_i)_{i\in I} \subseteq C_{\omega_1}(Y, f_i)_{i\in I}$. □

2. Polish spaces

Descriptive set theory is usually developed for Polish spaces i.e. topological spaces that are homeomorphic to some complete separable metric space. In this section we introduce some examples of these and study the connections between them, especially the connection between the n-dimensional Euclidian space $\mathbf{R}^n$, $n < \omega$, and the Baire space ω^ω (in Section 3, another connection is established). Later we usually develop the theory first for the Baire space and then use the connection(s) to translate (most of) the results to the Euclidian spaces, which are the most important polish spaces for the obvious reasons. Alternatively one can, of course, modify the proofs to get (most of) the results also for the Euclidian spaces. If the result does not generalize to Euclidian spaces, this is pointed out.

There are many reasons why we do not work directly with the Euclian spaces (or Polish spaces in general): E.g. the topology of Euclidian spaces is a bit obscure and all codings (and we will do a lot of them) are much more naturally done with functions from ω to ω than with reals - what ever the reals are.

In fact, we do not bother to specify what reals are: one can take any of the usual constructions of reals starting from natural number which we already have (e.g. Dedekind cuts of rational numbers $\mathbf{Q}$ or equivalence classes of Cauchy sequences in $\mathbf{Q}$, and $\mathbf{Q}$ is the field of fractions of the ring of integers etc.), and notice that the construction can be done in ZFC. Also the axiomatic approach to reals works in ZFC (completely ordered field with no infinitesimals).

Let us start with the reals $\mathbf{R}$: Open intervals $(q, r) \subseteq \mathbf{R}$, $q, r \in \mathbf{Q}$ and $q < r$, are called basic open sets. Then open sets are unions (of any size) of these basic open sets (notice that $\cup\emptyset = \emptyset$ and thus $\emptyset$ is open). This gives the same topology as the one, we get from the usual metric of $\mathbf{R}$.

If X is a topological space and $0 < n \leq \omega$, then by X^n we do not only mean the set X^n but the topological space we get from the set X^n by equipping it with the product topology in which open sets are unions of sets of the form $\Pi_{i<n}U_i = \{f : n \to \bigcup_{i<n} U_i |$ for every $i < n$, $f(i) \in U_i\}$, where each U_i is an open subset of X and (if $n = \omega$) all but finitely many of them are X. Notice that then the topology of $\mathbf{R}^n$, $n < \omega$, is the same as that one gets from the usual metric of $\mathbf{R}^n$.

2.1 Exercise. *Show that* $\mathbf{R}$ *and* $\mathbf{R}^2$ *are not homeomorphic. Hint: Suppose they are, take one element away from each of them and study the outcome.*

By $\mathbf{Ir}$ we denote the set $\mathbf{R} - \mathbf{Q}$ of irrational numbers. We equip it with the topology induced from $\mathbf{R}$ i.e. open sets are of the form $\mathbf{Ir} \cap U$ where U is an open subset of $\mathbf{R}$.

The Baire space $\mathbf{B}$ is the set ω^ω equipped with the following topology: Basic open sets are of the form $N_\eta = \{f \in \omega^\omega |\ \eta \subseteq f\}$, where η is a function from some $n < \omega$ to ω. Open sets are unions of these basic open sets. Notice that this topology is metrizable: The distance $d(f, g)$ between f and g from ω^ω is $1/(n+1)$ if n is the least natural number such that $f(n) \neq g(n)$. If there is no such n, then $f = g$ and the distance is 0. Clearly this is a metric, even an ultrametric i.e. $d(f, h) \leq max\{d(f, g), d(g, h)\}$ for any $f, g, h \in \omega^\omega$ (and the equality holds if $d(f, g) \neq d(g, h)$). It is easy to see that the topology we defined for ω^ω, is the same as the topology, one gets from this metric.

2.2 Lemma. *For all* $0 < n \leq \omega$, $\mathbf{B}^n$ *is homeomorphic with* $\mathbf{B}$.

Proof. We prove the claim in the case $n = \omega$, the other cases are similar. Let X_i, $i < \omega$ be as in Exercise 1.4.11 (ii). For all $i < \omega$, we define $f_i : \omega \to X_i$ as follows: $f_i(0)$ is the least element of X_i and $f_i(m + 1)$ is the least element of X_i strictly greater than $f_i(m)$. Then f_i is an order preserving function onto X_i. We denote the element $f_i(m)$ by a^i_m. Notice that for every $x \in \omega$, there are unique i and m such that $x = a^i_m$.

Now we can define a homeomorphism $F : \mathbf{B}^\omega \to \mathbf{B}$: For every $g = (g_i)_{i<\omega} \in \mathbf{B}^\omega$, we let $F(g) = h$ if $h : \omega \to \omega$ is such that for all $i, m < \omega$, $h(a^i_m) = g_i(m)$. Notice that then for all $i < \omega$, $g_i = h \circ f_i$.

2.2.1 Exercise. *Show that* F *is one-to-one and onto.*

We show that F is continuous, to show that F^{-1} is continuous is left as an exercise. Let N_η, $\eta : m \to \omega$, be a basic open set and $h \in N_\eta$. It is enough to find an open set U from $\mathbf{B}^\omega$ such that $h \in F[U] \subseteq N_\eta$ (because F is one-to-one). Pick $k < \omega$ such that if $i \geq k$, then $a^i_j \geq m$ for all $j < \omega$ (i.e. $X_i \cap m = \emptyset$ for all $i \geq k$). Notice that also for $i < k$, if $j \geq m$, then $a^i_j \geq m$ (f_i was order-preserving). Then for all $i < k$, choose $\eta_i : m \to \omega$ so that $\eta_i(j) = h(a^i_j)$ for all $j < m$. Now we let $U = \Pi_{i<\omega}U_i$, where $U_i = N_{\eta_i}$ if $i < k$ and otherwise $U_i = \omega^\omega$. Then U is as wanted (exercise). □

When we work in $\mathbf{B}^n$ for some $1 < n < \omega$ and want to apply results proved for $\mathbf{B}$ (which we can by Lemma 2.2) we should take as basic open sets the set $F(N_\eta)$, $\eta \in \omega^{<\omega}$, where F is the homeomorphism constructed above. All these are of the form $N_\eta = \{f \in \mathbf{B}^n |\ \eta(i) \subseteq f(i)$ for all $i < n\}$ where η is a function from n to $\omega^{<\omega}$. But all the sets of this form are not images of basic open sets of $\mathbf{B}$ under F. However, this does not really matter and so we take all of these as basic open sets. (If one is not happy with this, then instead of $(\omega^{<\omega})^n$, one should work with a suitable subset of it.)

The following lemma can be proved also by using continued fractions.

2.3 Lemma. *The Baire space is homeomorphic with* $\mathbf{Ir}$.

Proof. Let f be a one-to-one function from ω to $\mathbf{Q}$ such that $f(0) = 0$. We denote $f(n)$ by q^*_n. We also choose a one-to-one function g from ω onto $\mathbf{Z}$. At least half of the time in this proof we think that ω "is" $\mathbf{Z}$ (via this function). In particular, for every $0 < n < \omega$ and $\xi \in \omega^n$, by ξ^+ we mean the element of ω^n such that $\xi^+ \restriction (n-1) = \xi \restriction (n-1)$ and $g(\xi^+(n-1)) = g(\xi(n-1)) + 1$.

By induction on $0 < n < \omega$, we define rational numbers q_η for all $\eta \in \omega^n$ as follows (this definition may look complicated although what we do here is very simple; in case of difficulties, draw a picture):

$n = 1$: $q_\eta = g(\eta(0))$.

$n = m+1$ (and $m > 0$): We choose q_η, $\eta \in \omega^n$, so that the following holds: For every $\xi \in \omega^m$, let I_ξ be the open interval (q_ξ, q_{ξ^+}). Then the requirements are:

(i) For all $\eta \in \omega^n$, $q_\eta \in \mathbf{Q} \cap I_{\eta\restriction m}$, $q_\eta < q_{\eta^+}$ and $q_{\eta^+} - q^\eta \le 2^{-m}$.

(ii) For every $\xi \in \omega^m$, $inf\{q_\eta |\ \eta \in \omega^n,\ \xi \subseteq \eta\} = q_\xi$ and $sup\{q_\eta |\ \eta \in \omega^n,\ \xi \subseteq \eta\} = q_{\xi^+}$.

(iii) $q^*_m \in \{q_\xi |\ \xi \in \omega^{\le n}\}$ (recall: $\omega^{\le n} = \bigcup_{m \le n} \omega^m$).

Now we are ready to define a homeomorphism $F : \omega^\omega \to \mathbf{Ir}$: For every $f \in \omega^\omega$ we let $F(f)$ be the unique element in $\bigcap_{0<n<\omega} I_{f\restriction n}$ (i.e. $F(f) = sup\{q_{f\restriction n} |\ 0 < n < \omega\} = inf\{q_{(f\restriction n)^+} |\ 0 < n < \omega\}$). Notice that $F(f) \notin \{q_\eta |\ \eta \in \bigcup_{n<\omega} \omega^n\}$ and thus by (iii) above, the case $n = 1$ and the fact that $g(0) = 0$, $F(f) \notin \mathbf{Q}$ and thus $F(f) \in \mathbf{Ir}$.

2.3.1 Exercise.
(i) Show that F is one-to-one and onto.
(ii) Show that F is continuous.
(iii) Show that F^{-1} is continuous.

□

2.4 Exercise.
(i) Show that $\mathbf{Ir}^n$, $0 < n \le \omega$, is a subspace of $\mathbf{R}^n$ (i.e. the topology of $\mathbf{Ir}^n$ is the same as that induced from $\mathbf{R}^n$) and conclude that as a subspace of $\mathbf{R}^n$, $\mathbf{Ir}^n$ is homeomorphic with the Baire space.
(ii) Show that $|\mathbf{R}| = 2^\omega$.

If we equip 2^ω with a topology much the same way we did in the case of the Baire space, we get the Cantor space $\mathbf{C}$. So the basic open sets are of the form $N^c_\eta = \{f \in 2^\omega |\ \eta \subseteq f\}$, where $\eta \in 2^n$ for some $n < \omega$. Again open sets are unions of these basic open sets. Notice that $\mathbf{C}$ is a subspace of $\mathbf{B}$.

2.5 Lemma. *The Cantor space is homeomorphic with a bounded and closed (i.e. compact) subspace of* $\mathbf{R}$.

Proof. This proof is very similar to the proof of Lemma 2.3, only much simpler: For every $\eta \in 2^n$, $0 < n < \omega$, we define q_η by induction on n as follows:

$n=1$: We let $q_\eta = 0$ if $\eta(0)=0$ and otherwise $q_\eta = 2/3$.

$n=m+1$ (and $m>0$): We let $q_\eta = q_{\eta\restriction m}$ if $\eta(m)=0$ and otherwise, $q_\eta = q_{\eta\restriction m} + 2/3^n$.

Then we define $F: 2^\omega \to \mathbf{R}$ so that $F(f) = sup\{q_{f\restriction n} \mid 0<n<\omega\}$.

Clearly $rng(F) \subseteq [0,1]$. It is also closed: For all $\eta \in 2^n$, $0<n<\omega$, by q^+_η we mean $q_\eta + 1/3^n$ and we let I_η be $[q_\eta, q^+_\eta]$. Now it is easy to see that $F(f)$ is the unique element of $\bigcap_{0<n<\omega} I_{f\restriction n}$ and thus $rng(F) = \bigcap_{0<n<\omega}(\cup_{\eta\in 2^n} I_\eta)$, in particular $rng(F)$ is closed.

2.5.1 Exercise.

(i) Show that F is one-to-one and continuous.

(ii) Show that if $C = rng(F)$ is equipped with the topology induced from $\mathbf{R}$, then $F^{-1}: C \to 2^\omega$ is continuous.

□

2.6 Exercise.

(i) Show that the Cantor space is not homeomorphic with the Baire space nor with $\mathbf{R}^n$ for any $0<n\le\omega$. Hint: Compactness.

(ii) Show that $\mathbf{B}$ is not homeomorphic with $\mathbf{R}$.

(iii) Show that $\mathbf{C}^2$ is homeomorphic with $\mathbf{C}$.

3. Borel sets and the property of Baire

In this section we define Borel sets and to get a feeling to the notion, prove one of their basic properties, namely that they have the property of Baire and one consequence of this.

For the rest of these notes, when we talk about a topological space without specifying the space, we have in our mind (a subspace of) one from the set $\{\mathbf{B},\mathbf{C}\}\cup\{\mathbf{R}^n, \mathbf{Ir}^n \mid 0<n<\omega\}$ (keep in mind that $\mathbf{Ir}^n$ is homeomorphic with $\mathbf{B}$). We talk about topological spaces in general mainly only in the definitions.

3.1 Definition. *Let S be a topological space. The class $Borel(S)$ of all Borel sets in S is the least collection of subsets of S which contains all open sets and is closed under complements, countable unions and countable intersections. If $S=\mathbf{B}$, we may omit it from the notation.*

3.2 Exercise.

(i) Show that $Borel(\mathbf{B})$ is the least collection of subsets of $\mathbf{B}$ which contains all open sets and is closed under complements and countable unions.

(ii) Show that $Borel(\mathbf{B})$ is the least collection of subsets of $\mathbf{B}$ which contains all open sets and is closed under countable unions and countable intersections.

(iii) Prove (ii) for $\mathbf{R}^2$ in place of $\mathbf{B}$.

(iv) Show that if $X, Y \in Borel(S)$, S a topological space, then $X\times Y \in Borel(S^2)$.

(v) Show that if S' is a subspace of a topological space S and $S' \in Borel(S)$, then $Borel(S') \subseteq Borel(S)$.

The following, so called Borel hierarchy, is studied more closely later but we introduce it already now, because it is convenient in some proofs.

3.3 Definition. *Let S be a topological space. We define classes $\Sigma_\alpha(S)$ and $\Pi_\alpha(S)$ for $0<\alpha<\omega_1$ as follows:*

(i) $\Sigma_1(S)$ is the class of all open sets and $\Pi_1(S)$ is the class of all closed sets,

(ii) For $\alpha>1$, $\Sigma_\alpha(S)$ is the class of all countable unions of sets from $\bigcup_{\beta<\alpha}\Pi_\beta(S)$ and $\Pi_\alpha(S)$ is the class of all countable intersections of sets from $\bigcup_{\beta<\alpha}\Sigma_\beta(S)$.

If $S=\mathbf{B}$, we may omit it from the notation so e.g. Σ_1 is the class of all open subsets of $\mathbf{B}$.

$\Pi_2(\mathbf{R}^n)$ sets are often called G_δ sets and $\Sigma_2(\mathbf{R}^n)$ sets are often called F_σ set.

3.4 Exercise.

(i) Show for $0<n<\omega$, that every countable subset of R^n is $\Sigma_2(\mathbf{R}^n)$.

(ii) Show that for all $X\subseteq\omega^\omega$ and $0<\alpha<\omega_1$, $X\in\Sigma_\alpha$ iff $\omega^\omega-X\in\Pi_\alpha$. Conclude that $\mathbf{Ir}$ is $\Pi_2(\mathbf{R})$.

(iii) Show that for all $0<\alpha<\omega$, $\Sigma_\alpha\subseteq\Sigma_{\alpha+1}$ (and thus for $0<\alpha<\beta<\omega_1$, $\Sigma_\alpha\subseteq\Sigma_\beta$).

(iv) Show that $Borel=\bigcup_{0<\alpha<\omega_1}\Sigma_\alpha$.

(v) Show that for all $0<\alpha<\omega_1$ Σ_α and Π_α are closed under finite unions and intersection.

(vi) Suppose that $X,Y\subseteq\mathbf{R}$ are $\Sigma_\alpha(\mathbf{R})$ ($\Pi_\alpha(\mathbf{R})$). Show that $X\times Y$ is $\Sigma_\alpha(\mathbf{R}^2)$ ($\Pi_\alpha(\mathbf{R}^2)$). Conclude that $\mathbf{Ir}^n$ is $\Pi_2(\mathbf{R}^n)$.

(vii) Let $1<\alpha<\omega_1$ and $0<n<\omega$. Show that if $X\subseteq\mathbf{R}^n$ is $\Pi_\alpha(\mathbf{R}^n)$, then $X\cap Ir^n$ is $\Pi_\alpha(\mathbf{Ir}^n)$ and that the opposite direction holds if $n=1$.

(viii) Show that $|Borel|=2^\omega$. Hint: $|2^\omega\times\omega_1|\le|2^\omega\times 2^\omega|\le|(2^\omega)^\omega|=2^\omega$.

(ix) Show, without using Theorem A.4, that there is a subset of ω^ω which is not Borel.

3.5 Definition. *Let S be a topological space. We say that a subset of S is co-meager if it contains a countable intersection of open and dense subsets of S. A subset of S is meager, if the complement of it is co-meager.*

The following lemma is known in the literature as Baire's theorem.

3.6 Lemma. *Every co-meager subset of $\mathbf{B}$ is dense.*

Proof. Suppose D_i, $i<\omega$, are open and dense. It is enough to show that $\bigcap_{i<\omega}D_i$ is dense. For this let η be a function from some $n<\omega$ to ω. It is enough to show that $N_\eta\cap\bigcap_{i<\omega}D_i\neq\emptyset$.

But it is easy to find for all $i<\omega$, $n_i<\omega$ and $\eta_i\in\omega^{n_i}$ such that for all $i<j<\omega$, $N_{\eta_i}\subseteq D_i$ and $\eta\subseteq\eta_i\subseteq\eta_j$. (E.g. since D_0 is dense, $N_\eta\cap D_0\neq\emptyset$. Let $f\in N_\eta\cap D_0$. Since D_0 is open, there is $m<\omega$ such that $N_{f\restriction m}\subseteq D_0$. Then $n_0=max\{n,m\}$ and $\eta_0=f\restriction n_0$ are as wanted.)

Then $f=\cup_{i<\omega}\eta_i\in N_\eta\cap\bigcap_{i<\omega}D_i$, □

3.7 Exercise.

(i) Show that $\mathbf{Ir}^n$ is co-meager in $\mathbf{R}^n$ and that $X\subseteq\mathbf{R}^n$ is co-meager in $\mathbf{R}^n$ iff $X\cap\mathbf{Ir}^n$ is co-meager in $\mathbf{Ir}^n$.

(ii) Show that the family CM of all co-meager subsets of $\mathbf{B}$ is closed under countable intersections. Conclude that the family M of all meager sets is a σ-ideal

i.e. it is closed under countable unions, $\mathbf{B} \notin M$*,* $\emptyset \in M$*, and if* $X \subseteq Y \in M$*, then* $X \in M$*.*

(iii) Let $X \subseteq \mathbf{R}^2$ *be closed. Show that the boundary of* X *is meager. Conclude that there is an open set* $U \subseteq \mathbf{R}^2$ *such that* $X \Delta U = (X - U) \cup (U - X)$ *is meager.*

3.8 Definition. *We say that a subset of a topological space has the property of Baire (PB) if there is an open set* $U \subseteq \mathbf{B}$ *such that* $X \Delta U$ *is meager.*

3.9 Exercise.

(i) Show that a set A *has PB iff there is an open set* U *and a co-meager set* X *such that* $A \cap X = U \cap X$*.*

(ii) Show that $X \subseteq \mathbf{R}$ *has PB (in* $\mathbf{R}$*) iff* $X \cap \mathbf{Ir}$ *has PB (in* $\mathbf{Ir}$*).*

3.10 Lemma. *Every Borel set* $X \subseteq \mathbf{B}$ *has PB. In fact, the family of all subsets of* $\mathbf{B}$ *that have PB is a* σ*-algebra i.e. the family is closed under complements, countable unions and countable intersections.*

Proof. Clearly every open set has PB. Also closed sets C have PB. Let $int(C)$ be the set of those $f \in \mathbf{B}$ such that for some $n < \omega$, $N_{f \restriction n} \subseteq C$. Clearly $int(C)$ is open and $C - int(C)$ is meager (see, Exercise 3.7 (iii)). And thus C has PB. But from this we get more. For every $X \subseteq \mathbf{B}$, if there is a closed C such that $X \Delta C$ is meager, then X has PB: $int(C)$ witnesses this since $X \Delta int(C) \subseteq (X - C) \cup (C - int(C)) \cup (C - X)$. Thus if X has PB, so does $\mathbf{B} - X$.

So as in Exercise 3.2 (ii), it is enough to show that the set of subsets of $\mathbf{B}$ that have PB is closed under countable unions. But this is clear by Exercise 3.7 (ii), since if X_i, $i < \omega$, have PB witnessed by U_i, $i < \omega$, then $U = \cup_{i<\omega} U_i$ is open and $(\bigcup_{i<\omega} X_i) \Delta U \subseteq \bigcup_{i<\omega} (X_i \Delta U_i)$. □

3.11 Definition. *Suppose* S *and* S' *are topological spaces. We say that* $f : S \to S'$ *is Borel (aka Borel measurable) if* $f^{-1}[U]$ *is Borel for every open set* $U \subseteq S'$*.*

3.12 Exercise.

(i) Let $f : \mathbf{B} \to \mathbf{B}$*. Show that the following are equivalent.*

(a) f *is Borel.*

(b) $f^{-1}[N_\eta]$ *is Borel for every* $\eta \in \omega^{<\omega}$*.*

(c) $f^{-1}[X]$ *is Borel for every Borel set* $X \subseteq \mathbf{B}$*.*

(ii) Suppose S *and* S' *are topological spaces (in particular,* S' *is, say,* $\mathbf{Ir}$*,* $\mathbf{R}^n$ *or* $\mathbf{C}$*) and* $f, g : S \to S'$ *are Borel. Show that* $(x, y) \mapsto (f(x), g(y))$ *is a Borel function from* $S \times S$ *to* $S' \times S'$*.*

(iii) Suppose S*,* S' *and* S'' *are topological spaces and* $f : S \to S'$ *and* $g : S' \to S''$ *are Borel. Show that* $g \circ f$ *is Borel.*

3.13 Corollary. *If* $f : \mathbf{B} \to \mathbf{B}$ *is Borel, then it is continuous in a co-meager set i.e. there is a co-meager* $X \subseteq \mathbf{B}$ *such that* $f \restriction X$ *is continuous (when the topology of* X *is that induced from* $\mathbf{B}$*).*

Proof. For all $n < \omega$ and $\eta \in \omega^n$, we construct co-meager sets Y_η, X_n and open sets U_η as follows:

(i) $n = 0$: We let $Y_\emptyset = X_0 = U_\emptyset = \mathbf{B}$.

(ii) $n > 0$: For all $\eta \in \omega^n$, by PB, let Y_η be a co-meager set and U_η an open set such that $Y_\eta \cap U_\eta = Y_\eta \cap f^{-1}[N_\eta]$. Then we let $X_n = \bigcap_{\eta \in \omega^n} Y_\eta$.

Notice that for all $\eta \in \omega^n$ and $X \subseteq Y_\eta$, $(f \restriction X)^{-1}[N_\eta]$ is open in X (in the induced topology witnessed by U_η) and that $X_n \subseteq Y_\eta$ is co-meager. But then clearly $X = \bigcap_{n<\omega} X_n$ is the wanted co-meager set. □

We finish this section by proving a lemma that gives further information on the connection between $\mathbf{B}$ and $\mathbf{R}^n$. We do not prove the best possible result (one can choose X, Z and f so that $X = Z$), only one that is good enough for Section 10 where it is needed.

3.14 Lemma. *For all $0 < n < \omega$, there are a closed subspace Z of $\mathbf{B}$, continuous $f : Z \to \mathbf{R}^n$ and a Borel set $X \subseteq Z$ such that $f \restriction X$ is a one-to-one function from X onto $\mathbf{R}^n$ and $(f \restriction X)^{-1}$ is Borel.*

Proof. We prove this for $n = 1$, the other cases are similar. Let q_i, $i < \omega$, be an enumeration of $\mathbf{Q}$. We let Z be the set of those $\eta \in \mathbf{B}$ such that for all $i < j < \omega$, $|q_{\eta(i)} - q_{\eta(j)}| \leq 2^{-i} + 2^{-j}$.

3.14.1 Exercise.
(i) Z is closed.
(ii) For all $\eta \in Z$, ($(q_{\eta(i)})_{i<\omega}$ is a Cauchy sequence and thus) $f(\eta) = lim_{i\to\infty} q_{\eta(i)}$ exists.
(iii) For all $\eta \in Z$ and $i < \omega$, $|f(\eta) - q_{\eta(i)}| \leq 2^{-i}$.
(iv) If $f : Z \to \mathbf{R}$ is continuous.

For all $r \in \mathbf{R}$, let $\eta_r \in \mathbf{B}$ be such for all $j < \omega$, $\eta_r(j)$ is the least $i < \omega$ such that $|r - q_i| < 2^{-j}$. Notice that then $\eta_r \in Z$ and $f(\eta_r) = r$. Let $X = \{\eta_r | \, r \in \mathbf{R}\}$. Clearly $f \restriction X$ is one-to-one and onto $\mathbf{R}$.

3.14.2 Exercise. *Show that for all open $U \subseteq \mathbf{B}$, $f(U \cap X)$ is Borel in $\mathbf{R}$. Hint: Show by induction on $dom(\eta)$, that $f(N_\eta \cap X)$ is a (very simple) Borel set for all $\eta \in \omega^{<\omega}$.*

So we are left to prove that X is Borel. Clearly, $X = X_0 \cap X_1$, where

(1) X_0 is the set of those $\eta \in Z$ such that for all $i < \omega$ there are $m < \omega$ and $i < j < \omega$ such that for all $j < k < \omega$, $|q_{\eta(i)} - q_{\eta(k)}| \leq 2^{-i} - 1/m$.

(2) X_1 is the set of those $\eta \in Z$ such that for all $n, m, i < \omega$ there is $i < j < \omega$ such that for all $j < k < \omega$, if $n < \eta(i)$, then $|q_n - q_{\eta(k)}| > 2^{-i} - 1/m$.

If both of these sets are Borel, then X is Borel. As an example we show that X_0 is Borel, the other case is similar.

$$X_0 = \bigcap_{i<\omega} \bigcup_{m<\omega} \bigcup_{i<j<\omega} \bigcap_{j<k<\omega} \bigcup_{\xi \in X_{i,m,k}} (N_\xi \cap Z),$$

where $X_{i,m,k}$ is the set of those $\xi : k+1 \to \omega$ such that $|q_{\xi(i)} - q_{\xi(k)}| \leq 2^{-i} - 1/m$. □

3.15 Exercise. *Show that if $f : \mathbf{R}^m \to \mathbf{R}^n$ is Borel, then it is continuous in a co-meager set. Hint: Lemma 3.14 together with Corollary 3.13.*

4. Borel* codes and the perfect set property

In this section we introduce Borel* sets. It may look that we are making a simple thing complicated but in fact we do the opposite. Borel* codes will turn out to be an important tool in our studies of Borel (and projective) sets. To demonstrate the use of Borel* codes, in this section we will prove that continuum hypotheses is true for Borel sets i.e. they are either countable or of the same size as continuum (i.e. $\mathbf{R}$).

Usually trees are non-empty, but, for some technical reasons, below we do not require this.

4.1 Definition. *Let X be a non-empty set.*

(i) A (often non-empty) subset $T \subseteq X^{<\omega}$ is called a tree if for all $f \in T$ with $n = dom(f) > 0$ and for all $m < n$, $f \restriction m \in T$.

(ii) A non-empty tree $T \subseteq X^{<\omega}$ is called an ω-tree if the following holds:

(a) If $f : n \to X$ is in T and $n > 0$, then for all $x \in X$, $f \restriction (n-1) \cup \{(n-1, x)\} \in T$.

(b) There is no $f : \omega \to X$ such that for all $n < \omega$, $f \restriction n \in T$ (such a function is called an ω-branch of T).

(iii) We partially order trees T by $\subseteq$, call maximal elements leafs and denote by $L(T)$ the set of leaves. The least element (i.e. $\emptyset$) of T is called the root of T and if $f : n \to X$ is an element of T and not a root, then by f^- we denote the immediate predecessor of f i.e. $f \restriction (n-1)$. The elements of T are often called also nodes.

(iv) A Borel code is a pair (T, π), where $T \subseteq (\omega \times \omega)^{<\omega}$ is an ω-tree and π is a function from $L(T)$ to the basic open sets of $\mathbf{B}$.*

(v) For a Borel code $c = (T, \pi)$ and $\eta \in \mathbf{B}$, the Borel* game $GB^*(\eta, c)$ has two players, I and II and the game is played as follows: At each move $n < \omega$, a function $f_n : n+1 \to (\omega \times \omega)$ from T is chosen as follows: Suppose f_{n-1} is chosen ($f_{-1} = \emptyset$). If f_{n-1} is not a leaf of T, then first I chooses some $i < \omega$ and then II chooses $j < \omega$. This determines $f_n = f_{n-1} \cup \{(n-1, (i,j))\}$. If f_{n-1} is a leaf, then the game ends and II wins if $\eta \in \pi(f_{n-1})$.*

(vi) A function $W : \omega^{<\omega} \to \omega$ is a winning strategy of II in $GB^*(\eta, (T, \pi))$, if II wins every game by choosing $W(i_0, ..., i_n)$ on every move n, where $i_0, ..., i_n$ are the moves I has made on moves $0, ..., n$.

(vii) A Borel code c is a Borel* code of $X \subseteq \mathbf{B}$, if for all $\eta \in \mathbf{B}$, $\eta \in X$ iff II has a winning strategy in $GB^*(\eta, c)$.*

(viii) A set $X \subseteq \mathbf{B}$ is a Borel set if it has a Borel* code.*

4.2 Theorem. *Every Borel set is a Borel* set.*

Proof. We start by showing that open sets are Borel*: Let $U \neq \emptyset$ be open and find $\eta_i \in \omega^{<\omega}$ such that $U = \bigcup_{j<\omega} N_{\eta_j}$. Then we let $T = (\omega \times \omega)^{\leq 1}$ and define π so that $\pi(\{(0,(i,j))\}) = N_{\eta_j}$. Clearly, (T, π) is a Borel* code for U. The case $U = \emptyset$ is left as an exercise.

Suppose then that for every $i < \omega$, a set X_i has a Borel* code (T_i, π_i). We show that $X = \bigcap_{i<\omega} X_i$ has a Borel* code. The proof for the union goes similarly and is left as an exercise.

We let T be the set of those $f : n \to (\omega \times \omega)$, $n < \omega$, which satisfy: If $n > 0$ and $f(0) = (i,j)$, then there is $g : n-1 \to (\omega \times \omega)$ in T_i such that for all

$0 < m < n$, $f(m) = g(m-1)$. We define π so that for all $f \in L(T)$, $\pi(f) = \pi_i(g)$ if $f(0) = (i,j)$ for some $j < \omega$ and $g \in T_i$ is such that for all $0 < m < dom(f)$, $g(m-1) = f(m)$.

We show that if $\eta \in X$, then II has a winning strategy W for $GB^*(\eta, (T,\pi))$. The other direction is left as an exercise. Since $\eta \in X$, $\eta \in X_i$ for all $i < \omega$. Thus for all $i < \omega$, there is a winning strategy W_i for II in $GB^*(\eta, (T_i, \pi_i))$. For all $i < \omega$, we let $W(i) = 0$ and for $n > 0$, we let $W(i_0, ..., i_n) = W_{i_0}(i_1, ..., i_n)$. Clearly this is a winning strategy for II. □

A closer study of the proof of Theorem 4.2 shows:

4.3 Corollary. *For every $\alpha < \omega_1$, there is an ω-tree T such that for all $X \in \Sigma_\alpha$, there is a function π from $L(T)$ to the basic open sets such that (T,π) is a Borel* code for X.* □

4.4 Exercise. *We define Borel** sets exactly as Borel* sets were defined except that we allow π to take any Borel sets as values. Show that Borel**=Borel*.*

By Exercise 4.4 above, when we prove that some set is Borel by giving it a Borel code (T,π), we may use any Borel sets as values of π, in particular π may take $\emptyset$ as a value.

In the next definition we define ranks also for elements that are not nodes of the tree. Usually this is not done but it is convenient in Section 10 and thus we do it.

4.5 Definition. *Suppose X is a non-empty set and $T \subseteq X^{<\omega}$ is a tree.*

(i) If T does not have ω-branches, i.e there is no $f : \omega \to X$ such that for all $n < \omega$, $f \restriction n \in T$, we define a rank $rk(\eta) = rk(\eta; T) \in On \cup \{-1\}$ for all $\eta \in X^{<\omega}$ as follows:

(a) if $\eta \notin T$, then $rk(\eta) = -1$

(b) if $\eta \in T$ is a leaf, then $rk(\eta) = 0$,

(c) if $\eta \in T$ is not a leaf, then $rk(\eta) = \cup\{rk(\xi) + 1 |\ \xi^- = \eta\}$.

(ii) If T is non-empty and does not have ω-branches, then the rank of the root of T is called also the rank of T and is denoted by $rk(T)$. If T is empty, then we write $rk(T) = -1$ and if T has an ω-branch, we say that the rank of T is ∞. We also use convention that $-1 < \alpha$ and $\infty > \alpha$ for all ordinals α.

4.6 Exercise.

(i) Show that if X is non-empty and countable and $T \subseteq x^{<\omega}$ is a tree without ω-branches, then every node of T has a rank and that it is $< \omega_1$. Hint: Notice that $\eta < \xi$ if $\xi \subsetneq \eta$ is a well-founded partial order on T.

(ii) Show that every Borel set is Borel. Hint: Prove this by induction on $rk((T,\pi)) = rk(T)$.*

Now we have tools to prove that continuum hypothesis is true for Borel sets. In fact we prove more.

4.7 Definition. *We say that a subset X of a topological space S is perfect if with the induced topology it is homeomorphic to the Cantor space $\mathbf{C}$.*

4.8 Theorem. *Every uncountable Borel$(\mathbf{B})$ set contains a perfect set (and thus Borel sets are either countable or of the same size as $\mathbf{R}$).*

Proof. Let X be an uncountable Borel set. By Theorem 4.2 find a Borel* code (T,π) for it. Let $h:\omega^{<\omega}\rightarrow\omega$ be one-to-one and onto. For every $f\in\omega^\omega$, let $W_f:\omega^{<\omega}\rightarrow\omega$ be the following strategy of II in the games $GB^*(.,(T,\pi))$: $W_f(i_0,...,i_n)=f(h(i_0,...,i_n))$. Let P be the set of $(\eta,f)\in\omega^\omega\times\omega^\omega$ such that W_f is a winning strategy of II in $GB^*(\eta,(T,\pi))$. For all $R\subseteq P$, by $pr(R)$ we mean the set of those $\eta\in\omega^\omega$ for which there is $f\in\omega^\omega$ such that $(\eta,f)\in R$ and for $u=(a,p)\in(\omega^n\times\omega^n)$, $n<\omega$, by $R(u)$ we mean the set of those $(\eta,f)\in R$ for which $a\subseteq\eta$ and $p\subseteq f$.

4.8.1 Exercise. *Show that P is closed.*

For all ordinals α we define sets P_α as follows:

(a) $P_0=P$,

(b) $P_{\alpha+1}=P_\alpha-P^*_\alpha$ where P^*_α is the union of all $P_\alpha(u)$, $u\in(\omega^n\times\omega^n)$, $n<\omega$, such that $pr(P_\alpha(u))$ contains at most one element,

(c) for limit α, $P_\alpha=\bigcap_{\gamma<\alpha}P_\gamma$.

4.8.2 Exercise.

(i) Show that there is $\alpha^*<\omega_1$ such that $P_{\alpha^*+1}=P_{\alpha^*}$.

(ii) Show that for all α, $pr(P^*_\alpha)$ is countable.

(iii) Show that P_{α^*} is closed.

We show that $P_{\alpha^*}\neq\emptyset$. For a contradiction, suppose $P_{\alpha^*}=\emptyset$. For every $\eta\in X$, choose f_η so that $(\eta,f_\eta)\in P$. Since $P_{\alpha^*}=\emptyset$, for all $\eta\in X$, there is $\gamma_\eta<\alpha^*$ such that $(\eta,f_\eta)\in P^*_\gamma$. Since $\alpha^*<\omega_1$ and X is uncountable, there is uncountable $Y\subseteq X$ and $\gamma<\alpha^*$, such that for all $\eta\in Y$, $(\eta,f_\eta)\in P^*_\gamma$. This contradicts Exercise 4.8.1 (ii).

Now we are ready to construct a a perfect set $C\subseteq X$. For all $\xi\in 2^{<\omega}$ we define $\eta_\xi,f_\xi\in\omega^\omega$ and $n_\xi<\omega$ as follows:

(I) $(\eta_\emptyset,f_\emptyset)$ is any element of P_{α^*} and $n_\emptyset=0$.

(II) Suppose $\eta_\xi,f_\xi\in\omega^\omega$ and $n_\xi<\omega$ have been chosen for $\xi\in 2^n$. We choose them for $\xi_0=\xi\cup\{(n,0)\}$ and $\xi_1=\xi\cup\{(n,1)\}$ as follows: We let $\eta_{\xi_0}=\eta_\xi$ and $f_{\xi_0}=f_\xi$. By the choice of α^*, $P_{\alpha^*}((\eta_\xi\restriction n_\xi,f_\xi\restriction n_\xi))$ must contain some (η_{ξ_1},f_{ξ_1}) such that $\eta_{\xi_1}\neq\eta_{\xi_0}$ $(=\eta_\xi)$. We let $n_{\xi_0}=n_{\xi_1}=n<\omega$ be such that $\eta_{\xi_0}\restriction n\neq\eta_{\xi_1}\restriction n$.
For all $\xi\in 2^\omega$, we let $\eta_\xi=\bigcup_{k<\omega}\eta_{\xi\restriction k}\restriction n_{\xi\restriction k}$ and $f=\bigcup_{k<\omega}f_{\xi\restriction k}\restriction n_{\xi\restriction k}$. Since P_{α^*} is closed, for all $\xi\in 2^\omega$, $(\eta_\xi,f_\xi)\in P_{\alpha^*}\subseteq P$ and thus $\eta_\xi\in X$. It is easy to see that $C=\{\eta_\xi|\ \xi\in 2^\omega\}$ is perfect ($\xi\mapsto\eta_\xi$ is the required homeomorphism). □

4.9 Exercise. *(i) (Cantor-Bendixson theorem) For every $Y\subseteq\mathbf{B}$, let the derivative Y' of Y be the set of all non-isolated points of Y. Now suppose that $X\subseteq\mathbf{B}$ is uncountable and closed and define:*

(a) $X_0=X$,

(b) $X_{\alpha+1}=X'_\alpha$,

(c) for limit α, $X_\alpha=\bigcap_{\gamma<\alpha}X_\gamma$.

Let α^* be the least such that $X_{\alpha^*+1}=X_{\alpha^*}$. Show, without using Theorem 4.8, that (α^* exists and) X_{α^*} contains a perfect set.

(ii) Show that every uncountable $Borel(\mathbf{R}^n)$, $0<n<\omega$, set is either countable or contains a perfect set.

5. Universal relations and Borel hierarchy

In this section we study universal relations and as a corollary, we prove that the Borel hierarchy is proper.

5.1 Definition. *Let Γ be a collection of subsets of S for some topological space S We say that $R \subseteq (S \times S)$ is universal for Γ if for all $X \in \Gamma$, there is $\xi \in S$ such that $R_\xi = \{\eta \in S | \ (\eta, \xi) \in R\} = X$.*

5.2 Theorem. *For every $\alpha < \omega_1$, there is a Borel set $R \subseteq (\omega^\omega \times \omega^\omega)$ such that it is universal for Σ_α.*

Proof. Suppose $\alpha < \omega_1$ and let T be the ω-tree from Corollary 4.3 for this α. Let $h : L(T) \to \omega$ and $h' : \omega \to \omega^{<\omega}$ be one-to-one and onto. For every $f \in \omega^\omega$, let π_f be the function from $L(T)$ to the basic open sets such that for all $t \in L(T)$, $\pi_f(t) = N_{h'(f(h(t)))}$ (see the discussion after the proof of Lemma 2.2). Then by Corollary 4.3, for all $X \in \Sigma_\alpha$, there is $f \in \omega^\omega$ such that (T, π_f) is a Borel* code for X.

Now for all $t \in L(T)$ we let $\pi(t)$ be the set of all $(\eta, f) \in (\omega^\omega \times \omega^\omega)$ such that $\eta \in \pi_f(t)$. It is easy to see that $\pi(t)$ is open (exercise). We let $R \subseteq \mathbf{B} \times \mathbf{B}$ be the Borel set whose Borel* code (T, π) is. We show that R is as wanted.

Suppose $X \in \Sigma_\alpha$. Then there is $f \in \omega^\omega$ such that (T, π_f) is a Borel* code for X. We show that $X = R_f$. For this we show that $X \subseteq R_f$, the other inclusion is similar and is left as an exercise.

So suppose $\eta \in X$, We need to find a winning strategy for II in the game $GB^*((\eta, f), (T, \pi))$. Since (T, π_f) is a Borel* code for X, we know that II has a winning strategy W in the game $GB^*(\eta, (T, \pi_f))$. But by the definition of π, W is clearly also a winning strategy of II in the game $GB^*((\eta, f), (T, \pi))$. □

By going through the proofs of Lemma 4.2 and Theorem 5.2 carefully, one can see:

5.3 Corollary. *For every $0 < \alpha < \omega_1$, there is a Σ_α set $R \subseteq (\omega^\omega \times \omega^\omega)$ such that it is universal for Σ_α.* □

5.4 Exercise. *Show that for all $0 < \alpha < \omega_1$, the following are equivalent:*
(i) $\Sigma_\alpha = \Pi_\alpha$,
(ii) $\Sigma_{\alpha+1} = \Sigma_\alpha$,
(iii) $Borel = \Sigma_\alpha$.

5.5 Theorem. *For all $0 < \alpha < \omega_1$, $\Sigma_\alpha \neq \Pi_\alpha$.*

Proof. For a contradiction, suppose $\Sigma_\alpha = \Pi_\alpha$. Let R be as in Theorem 5.2 for this Σ_α. Then $X = \{\eta \in \mathbf{B} | \ (\eta, \eta) \notin R\}$ is Borel (exercise). Thus by Exercise 5.4 it is Σ_α and so there is $f \in \mathbf{B}$ such that for all $\eta \in \mathbf{B}$, $\eta \in X$ iff $(\eta, f) \in R$. So $f \in X$ iff $(f, f) \in R$ iff $f \notin X$, a contradiction. □

5.6 Exercise.

(i) Show that there is no Borel set $R \subseteq \mathbf{B} \times \mathbf{B}$ such that it is universal for Borel sets.

(ii) Show that if $R \subseteq \mathbf{B} \times \mathbf{B}$ is Borel, then for all $\eta \in \mathbf{B}$, $R_\eta = \{\xi \in \mathbf{B} | \ (\xi, \eta) \in R\}$ is Borel. Hint: Find a Borel code for R_η by modifying π from a Borel* code (T, π) of R.*

6. Projective hierarchy

We are mostly interested only about Σ^1_1 and Π^1_1 sets but for completeness and Section 14, we define the full hierarchy of projective sets already here. However, because of some technical difficulties, we define these first only in the Baire space $\mathbf{B}$ and then later for reals.

6.1 Definition.
(i) We let $\Sigma^1_0 = \Pi^1_0 = Borel(\mathbf{B})$,
(ii) $X \subseteq \mathbf{B}$ *is* Σ^1_{i+1} *if there is a* Π^1_i *set* $Y \subseteq \mathbf{B}\times\mathbf{B}$ *such that* $X = pr(Y) = \{\eta \in \mathbf{B} |$ *for some* $\xi \in \mathbf{B}$, $(\eta,\xi) \in Y\}$ *(keep in mind that* $\mathbf{B}\times\mathbf{B}$ *is homeomorphic with* $\mathbf{B}$ *and thus* Π^1_i *subsets of* $\mathbf{B}\times\mathbf{B}$ *have already been defined),*
(iii) $X \subseteq \mathbf{B}$ *is* Π^1_{i+1} *if* $\mathbf{B} - X$ *is* Σ^1_{i+1},
(iv) $X \subseteq \mathbf{B}$ *is* Δ^1_i *set if it is both* Π^1_i *and* Σ^1_i.
(v) Σ^1_1 *sets are often called analytic and* Π^1_1 *sets co-analytic.*
(vi) A set is called projective if it is Σ^1_i *set for some* $i < \omega$.

6.2 Exercise.
(i) Show that if $X \subseteq \mathbf{B}$ *is Borel, then there is a closed set* $Y \subseteq \mathbf{B}\times\mathbf{B}$ *such that* $X = pr(Y)$. *Conclude that Borel sets are* Δ^1_1. *Hint: See the proof of Theorem 4.8.*
(ii) Show that $\Sigma^1_n \cup \Pi^1_n \subseteq \Delta^1_{n+1}$ *for all* $n < \omega$.

6.3 Lemma. *The following are equivalent:*
(i) X *is* Σ^1_1,
(ii) $X = pr(Y)$ *for some closed* $Y \subseteq \mathbf{B}\times\mathbf{B}$,
(iii) $X = pr(Y)$ *for some* Σ^1_1 *set* $Y \subseteq \mathbf{B}\times\mathbf{B}$,
(iv) $X = f(Y)$ *for some closed set* $Y \subseteq \mathbf{B}$ *and continuous* $f : \mathbf{B} \to \mathbf{B}$,
(v) $X = \emptyset$ *or* $X = f[\mathbf{B}]$ *for some continuous* $f : \mathbf{B} \to \mathbf{B}$.
(vi) $X = \emptyset$ *or* $X = f(Y)$ *for some* Σ^1_1*-set* $Y \subseteq \mathbf{B}$ *and continuous* $f : Y \to \mathbf{B}$.

Proof. Clearly, (ii)$\Rightarrow$(i), (i)$\Rightarrow$(iii), (iv)$\Rightarrow$(ii) (since $\{(\eta,\xi) \in \mathbf{B}^2 |\ \xi \in Y,\ \eta = f(\xi)\}$ is closed) and (v)$\Rightarrow$(vi). So it is enough to show the following three implications:

(iii)$\Rightarrow$(iv): Suppose (iii) holds. Let $Z \subseteq \mathbf{B}^3$ be a Borel set such that Y is the set of those (η,ξ) such that for some ξ', $(\eta,\xi,\xi') \in Z$. Define $f : \mathbf{B}^3 \to \mathbf{B}$ so that $f(\eta,\xi,\xi') = \eta$. Since $\mathbf{B}^3$ is homeomorphic to $\mathbf{B}$ and f is continuous, (iv) follows from Exercise 6.2 (i) and the fact that the composition of two continuous functions is continuous.

(iv)$\Rightarrow$(v): By (iv), it is enough to show that if $C \subseteq \mathbf{B}$ is closed and non-empty then there is a continuous $f : \mathbf{B} \to \mathbf{B}$ such that $rng(f) = C$. For all $\xi \in \omega^n$, $n < \omega$, we choose $\eta_\xi \in \omega^n$ so that
(a) $\xi_\emptyset = \emptyset$,
(b) if $\xi \subseteq \xi'$, then $\eta_\xi \subseteq \eta_{\xi'}$,
(c) $N_{\eta_\xi} \cap C \neq \emptyset$,
(d) for all $g \in C$ and $\xi \in \omega^n$, $n < \omega$, if $g \in N_{\eta_\xi}$, then there is $\xi' \in \omega^{n+1} \cap N_\xi$ such that $g \in N_{\eta_{\xi'}}$.
By recursion on n it is easy to find these η_ξ. Then we define $f : \mathbf{B} \to \mathbf{B}$ so that for all $\xi \in \mathbf{B}$, $f(\xi) = \cup_{n<\omega}\eta_{\xi\restriction n}$. Clearly f is continuous, since C is

closed, (c) guarantees that $f(\xi) \in C$ for all $\xi \in \mathbf{B}$ and finally (d) guarantees that $C \subseteq rng(f)$.

(vi)$\Rightarrow$(iv): Suppose (vi) holds. If $X = \emptyset$, the claim is clear and thus we may assume that $X \neq \emptyset$. Since we have already seen that (i) implies (iv) and Y is Σ^1_1, there are a closed $Z \subseteq \mathbf{B}$ and continuous $g : \mathbf{B} \to \mathbf{B}$ such that $Y = g(Z)$. Now Z and $g \circ f$ witness that (iv) holds. $\square$

6.4 Exercise.

(i) Show that every uncountable Σ^1_1 set contains a perfect set. Hint: The proof of Theorem 4.8 works.

(ii) Show that Σ^1_1 sets are closed under countable unions and countable intersections. Hint: For intersections, start by showing that if $C_i \subseteq \mathbf{B}^2$ are closed, then so is the set of those $(\eta, (\xi_i)_{i<\omega}) \in \mathbf{B} \times \mathbf{B}^\omega$ such that for all $i < \omega$, $(\eta, \xi_i) \in C_i$.

(iii) Show that if $X, Y \subseteq \mathbf{B}$ are Σ^1_1, then so is $X \times Y \subseteq \mathbf{B} \times \mathbf{B}$.

(iv) Prove (ii) and (iii) above for Σ^1_n, $n > 1$.

(v) Suppose $f : \mathbf{B} \to \mathbf{B}$ is Borel. Show that if $X \subseteq \mathbf{B}$ is Σ^1_n, then $f^{-1}(X)$ is Σ^1_n.

(vi) Suppose $X \subseteq \mathbf{B}$ is Σ^1_n and $f; \mathbf{B} \to \mathbf{B}$ is continuous. Show that $f(X)$ is Σ^1_n.

Now we are ready to define $\Sigma^1_n(\mathbf{R}^m)$ and $\Pi^1_n(\mathbf{R}^m)$.

6.5 Definition. *Suppose $0 < m < \omega$ and $0 < n < \omega$. We say that $X \subseteq \mathbf{R}^m$ is $\Sigma^1_n(\mathbf{R}^m)$ if $X = f(Y)$ for some continuous $f : \mathbf{B} \to \mathbf{R}^m$ and Σ^1_n set $Y \subseteq \mathbf{B}$. X is $\Pi^1_n(\mathbf{R}^m)$ if $\mathbf{R}^m - X$ is $\Sigma^1_n(\mathbf{R}^m)$ and X is $\Delta^1_n(\mathbf{R}^m)$ if it is both $\Sigma^1_n(\mathbf{R}^m)$ and $\Pi^1_n(\mathbf{R}^m)$. If used, $\Sigma^1_0(\mathbf{R}^m) = \Pi^1_0(\mathbf{R}^m) = Borel(\mathbf{R}^m)$.*

We start by an exercise that shows that our notion of analytic sets in $\mathbf{R}^n$ is the usual one.

6.6 Lemma. *Suppose that $0 < n < \omega$ and $X \subseteq \mathbf{R}^n$.*

(i) If X is $Borel(\mathbf{R}^n)$, then it is $\Sigma^1_1(\mathbf{R}^n)$.

(ii) The following are equivalent.

(a) X is $\Sigma^1_1(\mathbf{R}^n)$.

(b) $X = \emptyset$ or $X = f(\mathbf{B})$ for some continuous $f : \mathbf{B} \to \mathbf{R}^n$.

(c) $X = pr(Y)$ for some $Borel(\mathbf{R}^n \times \mathbf{R})$ set $Y \subseteq \mathbf{R}^n \times \mathbf{R}$.

(d) $X = \emptyset$ or there are $0 < m < \omega$, a $Borel(\mathbf{R}^m)$ set $Y \subseteq \mathbf{R}^m$ and continuous $f : Y \to \mathbf{R}^n$ such that $X = f(Y)$.

(e) $X = \emptyset$ or there are $0 < m < \omega$, a $\Sigma^1_1(\mathbf{R}^m)$ set $Y \subseteq \mathbf{R}^m$ and continuous $f : Y \to \mathbf{R}^n$ such that $X = f(Y)$.

Proof. (i) is left as an exercise and we prove (ii):

The equivalence (a)$\Leftrightarrow$(b) and the implication (e)$\Rightarrow$(b) are clear by Lemma 6.3 (v). The implication (c)$\Rightarrow$(d) is immediate and (d)$\Rightarrow$(e) follows from (i).

We are left to prove the implication (b)$\Rightarrow$(c). Clearly we may assume that $X \neq \emptyset$. Then we notice that if f is as in (b) (equalizing $\mathbf{Ir}$ with $\mathbf{B}$), then $Y = \{(f(x), x) | \ x \in \mathbf{Ir}\}$ is a closed subset of $\mathbf{R}^n \times \mathbf{Ir}$. Thus it is a $Borel(\mathbf{R}^n \times \mathbf{R})$ subset of $\mathbf{R}^n \times \mathbf{R}$ ($Y = \overline{Y} \cap (\mathbf{R}^n \times \mathbf{Ir})$, where $\overline{Y}$ is the closure of Y in $\mathbf{R}^n \times \mathbf{R}$). Clearly, $X = pr(Y)$. $\square$

Next exercise lists some other basic properties of $\Sigma^1_i(\mathbf{R}^n)$ set. Most of the exercises can be solved simply by translating related results for Σ^1_i sets.

6.7 Exercise. *Suppose* $0 < n, m < \omega$.

(i) Show that $X \subseteq \mathbf{R}$ *is* $\Sigma^1_n(\mathbf{R})$ $(\Pi^1_n(\mathbf{R}))$ *iff* $X \cap \mathbf{Ir}$ *is* Σ^1_n (Π^1_n).

(ii) Show that $Borel(\mathbf{R}^m) \subseteq \Delta^1_1(\mathbf{R}^m)$.

(iii) Let $g = f \restriction X$ *be as in Lemma 3.14. Show that for all* $A \subseteq \mathbf{R}^n$ *and* $i < \omega$, A *is* $\Sigma^1_i(\mathbf{R}^n)$ $(\Pi^1_i(\mathbf{R}^n))$ *iff* $g^{-1}(A)$ *is* Σ^1_i (Π^1_i).

Hint: We prove the implication from left to right in the case $n = 1$*: If* A *is* $\Sigma^1_1(\mathbf{R}^n)$*, then there is closed* $C \subseteq \mathbf{R}^n \times \mathbf{Ir}$ *such that* $A = pr(C)$*. Then* C *is* $Borel(\mathbf{R}^n \times \mathbf{R})$ *and letting* $h : X \times \mathbf{Ir} \to \mathbf{R}^n \times \mathbf{R}$ *be* (g, id)*,* h *is continuous and thus* $h^{-1}(C)$ *is Borel in* $X \times \mathbf{Ir}$ *and so also Borel in* $\mathbf{B} \times \mathbf{Ir}$*, since* X *is Borel. Clearly* $g^{-1}(A)$ *is a projection of* $h^{-1}(C)$ *and so it is* Σ^1_1.

For the other direction, start by proving that there is a continuous $h : \mathbf{B} \to \mathbf{R}^n$ *such that* $h \restriction X = g$.

(iv) Show that $\Sigma^1_n(\mathbf{R}^m) \cup \Pi^1_n(\mathbf{R}^m) \subseteq \Delta^1_{n+1}(\mathbf{R}^m)$.

(v) Show that $\Sigma^1_n(\mathbf{R}^m)$ *is closed under countable unions and intersections.*

(vi) Show that every uncountable $\Sigma^1_1(\mathbf{R}^m)$ *set contains a perfect set.*

(vii) Show that if $Y \subseteq \mathbf{R}^n \times \mathbf{R}^m$ *is* $\Pi^1_i(\mathbf{R}^n \times \mathbf{R}^m)$*, then* $X = pr(Y) = \{x \in \mathbf{R}^n |$ *for some* $y \in \mathbf{R}^m$*,* $(x, y) \in Y\}$ *is* $\Sigma^1_{i+1}(\mathbf{R}^n)$.

The items (ii) and (iv) from Lemma 6.3 for $\mathbf{R}^n$ in place of $\mathbf{B}$ are not equivalent with being $\Sigma^1_1(\mathbf{R}^n)$. Next exercise (together with Exercise 6.2 and Theorem 5.5) explains why.

6.8 Exercise. *Show that if* $X \subseteq \mathbf{R}^2$ *is closed and* $f : \mathbf{R}^2 \to \mathbf{R}$ *is continuous (e.g.* $f = pr$*), then* $f(X)$ *is in* $\Sigma_2(\mathbf{R})$*. Hint: Show first that if in addition* X *is bounded (i.e. compact), then* $f(X)$ *is closed.*

We finish this section by studying universal relations.

6.9 Theorem. *There is a* Σ^1_1 *set* $R \subseteq \mathbf{B} \times \mathbf{B}$ *such that it is universal for* Σ^1_1.

Proof. Since $\Pi_1 \subseteq \Pi_2$, every closed set is a countable intersection of open sets. Fix a one-to-one function h from $\omega \times \omega$ onto ω and a one-to-one function h' from ω onto $\omega^{<\omega} \times \omega^{<\omega}$. Then every $\eta \in \mathbf{B}$ can be thought as a code for the set $C_\eta = \bigcap_{i<\omega} \bigcup_{j<\omega} N_{h'(\eta(h(i,j)))}$ (so $N_{h'(\eta(f(i.j)))}$ is a basic open set of $\mathbf{B}^2$). Thus for every closed $C \subseteq \mathbf{B} \times \mathbf{B}$, there is $\eta \in \mathbf{B}$ such that $C_\eta = C$. Also For every $\eta \in \mathbf{B}$, $pr(C_\eta)$ is Σ^1_1. Thus we can think every η also a code for the Σ^1_1 set $X_\eta = pr(C_\eta)$ and for every Σ^1_1 set $X \subseteq \mathbf{B}$, there is $\eta \in \mathbf{B}$ such that $X_\eta = X$.

Let S be the set of those $(\xi, \eta, \xi') \in \mathbf{B}^3$ such that $(\xi, \xi') \in C_\eta$ and R be the set of those $(\xi, \eta) \in \mathbf{B} \times \mathbf{B}$ such that for some ξ', $(\xi, \eta, \xi') \in S$. Clearly R is universal for Σ^1_1.

So it is enough to show that R is Σ^1_1. For this it is enough to show that S is Borel. But this is the case since, $S = \bigcap_{i<\omega} \bigcup_{j<\omega} U_{ij}$, where U_{ij} is the union of all $N_{(a,b,c)}$ such that $(a, b, c) \in (\omega^{<\omega})^3$, $dom(b) = h(i, j)+1$ and $(a, c) = h'(b(h(i, j)))$ (exercise). □

6.10 Exercise.

(i) Show that for all $0 < n < \omega$*, there is a* Σ^1_n *set* $R \subseteq \mathbf{B} \times \mathbf{B}$ *such that it is universal for* Σ^1_n*. Hint: Use induction and show first that if* R *is universal for* Σ^1_n*, then* $\mathbf{B}^2 - R$ *is universal for* Π^1_n.

(ii) Show that for all $0 < n < \omega$, $\Sigma^1_n \neq \Pi^1_n$. Conclude that for all $n < \omega$, $\Sigma^1_n \cup \Pi^1_n \subsetneq \Sigma^1_{n+1}$, in particular, $\Sigma^1_1 \neq Borel$. Hint: See the proof of Theorem 5.5.

(iii) Show that $\Sigma^1_1(\mathbf{R}^n) \neq Borel(\mathbf{R}^n)$.

(iv) Show that for all $0 < n < \omega$, there is a $\Sigma^1_n(\mathbf{R}^m)$ set $R \subseteq \mathbf{R}^m \times \mathbf{R}^m$ such that it is universal for $\Sigma^1_n(\mathbf{R}^m)$.

We finish this section with a 'geometric' exercise. Items (i) and (ii) in the exercise are hints for (iii).

6.11 Exercise.

(i) Let $A = \{(x_i)_{i<3} \in \mathbf{R}^3 |\ x_1 = x_2 = 0\}$. Show that there is a homeomorphism P from A (with the induced topology) onto $\mathbf{R}$ Conclude that for every $X \subseteq \mathbf{R}$, X is in $Borel(\mathbf{R})$ iff $P^{-1}[X]$ is in $Borel(\mathbf{R}^3)$.

(ii) Let $B = \{(x_i)_{i<3} \in \mathbf{R}^3 |\ x_1^2 + x_2^2 = 1, x_1 > 0\}$. Show that there is a homeomorphism H from B (with the induced topology) onto $\mathbf{R}^2$. Conclude that for all $X \subseteq \mathbf{R}^2$, X is in $Borel(\mathbf{R}^2)$ iff $H^{-1}[X]$ is in $Borel(\mathbf{R}^3)$.

(iii) Show that there is a $Borel(\mathbf{R}^3)$ set $X \subseteq \mathbf{R}^3$ such that $Y = \bigcup_{x \in X} \overline{B}(x,1)$ is not $Borel(\mathbf{R}^3)$, where $\overline{B}((x_i)_{i<3}, 1) = \{(y_i)_{i<3} \in \mathbf{R}^3 |\ (x_0 - y_0)^2 + (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq 1\}$. Hint: Use (i) and (ii) together with Exercise 6.10.

7. Separation

In this section we will show that if $X, Y \subseteq \mathbf{B}$ are Σ^1_1 and $X \cap Y = \emptyset$, then there is a Borel set Z such that $X \subseteq Z \subseteq (\mathbf{B} - Y)$.

We start by making simple observations on trees.

7.1 Definition. *Let $T, T' \subseteq X^{<\omega}$ be trees.*

(i) We write $T \leq T'$ if there is a function $f : T \to T'$ such that for all $\eta, \xi \in T$, if $\eta \subsetneq \xi$, then $f(\eta) \subsetneq f(\xi)$ (but not necessarily the other way round).

(ii) $GC(T, T')$ is the following game: At each move $n < \omega$, first player I chooses $t_n \in T$ such that $t_m \subsetneq t_n$ for all $m < n$ and then II chooses $t'_n \in T'$ such that $t'_m \subsetneq t'_n$ for all $m < n$. The first who can not move looses and if both can move on every round $n < \omega$, II wins. Winning strategy for II is defined as in the case of the game GB^*.

7.2 Exercise. *Let $T, T' \subseteq X^{<\omega}$ be trees.*

(i) $T \leq T'$ if there is a function $f : T \to T'$ such that for all $\eta, \xi \in T$, if $\eta \subsetneq \xi$ then $f(\eta) \subsetneq f(\xi)$ and in addition, for all $\eta \in T$, $dom(\eta) = dom(f(\eta))$.

(ii) $T \leq T'$ iff II has a winning strategy in $GC(T, T')$.

(iii) $T \leq T'$ iff $rk(T) \leq rk(T')$ (see Definition 4.5).

7.3 Theorem. *Suppose $X, Y \subseteq \mathbf{B}$ are Σ^1_1 and $X \cap Y = \emptyset$. Then there is a Borel set $Z \subseteq \mathbf{B}$ such that $X \subseteq Z \subseteq (\mathbf{B} - Y)$.*

Proof. Choose closed $X^*, Y^* \subseteq \mathbf{B} \times \mathbf{B}$ so that $X = pr(X^*)$ and $Y = pr(Y^*)$. If X^* or Y^* is empty, there is nothing to prove, and thus we assume that they are not empty.

For all $\eta \in \mathbf{B}$, let X_η be the set of all $\xi \in \omega^{<\omega}$ such that, letting $n = dom(\xi)$, there are $\eta', \xi' \in \mathbf{B}$ such that $(\eta', \xi') \in X^*$, $\eta' \restriction n = \eta \restriction n$ and $\xi \subseteq \xi'$. Y_η is defined similarly. Notice that for all $n < \omega$ and $\xi \in \omega^n$, whether $\xi \in X_\eta$ or not, depends on $\eta \restriction n$ only (and similarly for Y). Thus if $\eta, \xi \in \omega^n$, we write also

$\xi \in X_\eta$ meaning that for some $\eta' \in \mathbf{B}$, $\xi \in X_{\eta'}$ and $\eta \subseteq \eta'$ (and similarly for Y). Notice that for all $\eta \in \mathbf{B}$, $X_\eta = \bigcup_{n<\omega} X_{\eta\restriction n}$ (and similarly for Y).

7.3.1 Exercise.
(i) X_η and Y_η are trees.
(ii) $\eta \in X$ iff X_η contains an ω-branch i.e. there is $f \in \omega^\omega$ such that for all $n < \omega$, $f \restriction n \in X_\eta$. And similarly for Y.
(iii) If $\eta \in X$, then $Y_\eta \leq X_\eta$ and if $Y_\eta \leq X_\eta$, then $\eta \notin Y$.

So if we let Z be the set of those $\eta \in \mathbf{B}$ such that $Y_\eta \leq X_\eta$, then $X \subseteq Z \subseteq (\mathbf{B} - Y)$. Thus we are left to show that Z is Borel*.

The idea in showing that Z is Borel, is to find a Borel* code (T, π) so that for all $\eta \in \mathbf{B}$, the game $GB^*(\eta, (T, \pi))$ is, via coding, the same game as $GC(Y_\eta, X_\eta)$.

We let T be the set of all $f : n \to \cup_{m<\omega}(\omega^m)^3$ such that for all $i < n$ the following holds:
(a) $f(i) \in (\omega^i)^3$,
(b) if $j + 1 < n$ and $f(j) = (\xi_i)_{i<3}$, then $\xi_1 \in X_{\xi_0}$ and $\xi_2 \in Y_{\xi_0}$,
(c) if $j < k < n$, $f(j) = (\xi_i)_{i<3}$ and $f(k) = (\xi'_i)_{i<3}$, then for all $i < 3$, $\xi_i \subseteq \xi'_i$.

7.3.2 Exercise.
(i) Show that T is a tree.
(ii) Show that T does not have an ω-branch.

Now the tree T is almost as required in Borel* codes, except that the elements are not of the right form. It is easy to see that this can be fixed by an easy coding. However, for the clarity, we use T as it is in a Borel* code. So we need to explain how the players move: at each move $n < \omega$, again if $f_{n-1} : n \to \cup_{m<n}(\omega^m)^3$ from T is chosen, it is not a leaf and $f_{n-1}(n-1) = (\xi'_i)_{i<3}$ (if $n > 0$), then I chooses first some $\xi_2 \in \omega^n$ so that $\xi'_2 \subseteq \xi_2$ and then II chooses $(\xi_0, \xi_1) \in (\omega^n)^2$ so that $\xi'_i \subseteq \xi_i$ for $i < 2$. The play continues from $f_n = f_{n-1} \cup \{(n, (\xi_i)_{i<3})\} \in T$. We choose the labeling π as follows: Suppose $f : n+1 \to \cup_{m<n}(\omega^m)^3$ is a leaf. Notice that then $n > 0$. Let $f(n) = (\xi_i)_{i<3}$. If $\xi_2 \notin Y_{\xi_0}$, we let $\pi(f)$ be N_{ξ_0}, and otherwise we define $\pi(f) = \emptyset$.

7.3.3 Exercise. *Show that II has a winning strategy in $GB^*(\eta, (T, \pi))$ iff II has a winning strategy in $GC(Y_\eta, X_\eta)$. Conclude that (T, π) is a Borel* code for Z.*

□

7.4 Exercise. *Show that $\Delta^1_1 = Borel$ and $\Delta^1_1(\mathbf{R}^n) = Borel(\mathbf{R}^n)$.*

Next exercise gives a modified way of proving Theorem 7.3.

7.5 Exercise. *Let X, X^*, Y and Y^* be as in the beginning of the proof of Theorem 7.3. Let $h : \omega \to \omega^3$ be one-to-one and onto. For $f : n \to \omega$, $n \leq \omega$, let $f_k : n \to \omega$, $k < 3$, be such that for all $x < n$, if $h(f(x)) = (a_i)_{i<3}$, then $f_k(x) = a_k$. We let T be the set of all $f : n \to \omega$, $n < \omega$, such that there are $\eta, \eta', \xi, \xi' \in \mathbf{B}$ for which $f_0 \subseteq \eta$, $f_0 \subseteq \eta'$, $f_1 \subseteq \xi$, $f_2 \subseteq \xi'$, $(\eta, \xi) \in X^*$ and $(\eta', \xi') \in Y^*$. Let T' be the set of all $f : n \to \omega$, $n < \omega$, such that if $n > 0$,*

$f(0) = 0$ and $f^- \in T$, where $f^- : n-1 \to \omega$ is such that for all $x < n-1$, $f^-(x) = f(x+1)$.

(i) Show that T, and thus T', does not contain an ω-branch.

(ii) $T' \not\leq T$.

(iii) If $\eta \in X$, then $T' \leq X_\eta$ and if $T' \leq X_\eta$, then $T' \not\leq Y_\eta$.

(iv) $Z = \{\eta \in \mathbf{B} | \ T' \leq X_\eta\}$ separates X and Y (i.e. $X \subseteq Z \subseteq \mathbf{B} - Y$).

(v) Z is Borel.

7.6 Exercise. *Suppose $f : \mathbf{B} \to \mathbf{B}$ is such that $graph(f) = \{(x, f(x)) | \ x \in \mathbf{B}\}$ is Borel. Show that f is Borel (i.e. Borel-measurable).*

7.7 Exercise. *Suppose $f : \mathbf{B} \to \mathbf{B}$ is continuous, $A \subseteq \mathbf{B}$ is closed and $f \restriction A$ is one-to-one.*

(i) Suppose $n < \omega$. Show that for all $\eta \in \omega^n$, there are pairwise disjoint Borel sets B_η such that $f(N_\eta \cap A) \subseteq B_\eta \subseteq \overline{f(N_\eta \cap A)}$.

(ii) Show that in (i), we can choose the sets B_η so that if $\eta \subseteq \xi$, then $B_\xi \subseteq B_\eta$.

(iii) Show that $f(A)$ is Borel.

In fact, Exercise 7.7 (iii) is a characterization of being Borel, i.e. it is also true that if $X \subseteq \mathbf{B}$ is Borel then there are f and A as above so that $X = f(A)$.

8. Suslin operation

In this section we look at the way of defining Σ^1_1 sets that was used (by M. Suslin) when Σ^1_1 sets were introduced the first time (in 1917).

8.1 Definition. *Let S be a topological space and that for all $\eta \in \omega^{<\omega}$, $X_\eta \subseteq S$. Then by $\mathcal{A}\{X_\eta | \ \eta \in \omega^{<\omega}\}$ we mean the set of those $\xi \in S$ for which there is $f : \omega \to \omega$ such that $\xi \in \bigcap_{n<\omega} X_{f\restriction n}$.*

8.2 Exercise. *Suppose that for all $\eta \in \omega^{<\omega}$, $X_\eta \subseteq \mathbf{B}$ is Borel and that for all $\eta, \xi \in \omega^n$ if $\eta \neq \xi$, then $X_\eta \cap X_\xi = \emptyset$. Show that $\mathcal{A}\{X_\eta | \ \eta \in \omega^{<\omega}\}$ is Borel.*

8.3 Lemma. *If every X_η, $\eta \in \omega^{<\omega}$ is Σ^1_1, then so is $\mathcal{A}\{X_\eta | \ \eta \in \omega^{<\omega}\}$.*

Proof. For all $\eta \in \omega^{<\omega}$, let $Y_\eta = X_\eta \times N_\eta$. It is easy to see that Y_η is Σ^1_1 (exercise). Thus by Exercise 6.4 (iii), for all $n < \omega$, $Y_n = \bigcup_{\eta \in \omega^n} Y_\eta$ is still Σ^1_1. Thus still by Exercise 6.4 (iii), also $Y = \cap_{n<\omega} Y_n$ is Σ^1_1. But clearly $\mathcal{A}\{X_\eta | \ \eta \in \omega^{<\omega}\} = pr(Y)$ and thus, by Lemma 6.3, $\mathcal{A}\{X_\eta | \ \eta \in \omega^{<\omega}\}$ is Σ^1_1. □

8.4 Lemma. *If $X \subseteq \mathbf{B}$ is Σ^1_1, then there are closed sets $X_\eta \subseteq \mathbf{B}$, $\eta \in \omega^{<\omega}$, such that $X = \mathcal{A}\{X_\eta | \ \eta \in \omega^{<\omega}\}$.*

Proof. If $X = \emptyset$, the claim is clear. So we may assume that $X \neq \emptyset$. Then by Lemma 6.3, there is continuous $F : \mathbf{B} \to \mathbf{B}$ such that $X = F[\mathbf{B}]$. If $Y \subseteq \mathbf{B}$, by $\overline{Y}$ we denote the closure of Y. If $\eta, \xi \in \mathbf{B}$ and $\xi \neq F(\eta)$, there is $n < \omega$ such that $\xi \restriction n \neq F(\eta) \restriction n$. By continuity of F, there is $m < \omega$ such that $F[N_{\eta\restriction m}] \subseteq N_{F(\eta)\restriction n}$. Since $N_{F(\eta)\restriction n}$ is closed, also $\overline{F[N_{\eta\restriction m}]} \subseteq N_{F(\eta)\restriction n}$ and thus $\xi \notin \overline{F[N_{\eta\restriction m}]}$. So $\bigcap_{m<\omega} \overline{F[N_{\eta\restriction m}]} = \{F(\eta)\}$. Thus $X = \mathcal{A}\{X_\eta | \ \eta \in \omega^{<\omega}\}$ by letting X_η be $\overline{F[N_\eta]}$ for all $\eta \in \omega^{<\omega}$. □

8.5 Exercise.
(i) Prove that Lemma 8.3 holds for $\mathbf{R}^n$ in place of $\mathbf{B}$.
(ii) Check that the proof of Lemma 8.4 works also for $\mathbf{R}^n$ in place of $\mathbf{B}$.

The proof of the following lemma demonstrates the use of Suslin operation. The lemma itself will be needed in Section 10.

8.6 Lemma. *Let $T \subseteq \mathbf{B}^2$ be closed and for all $\eta \in \mathbf{B}$, define T_η as X_η was defined from X^* in the beginning of the proof of Theorem 7.3. Let $R \subseteq \mathbf{B}^2$ be the set of those (η, ξ) such that $T_\eta \leq T_\xi$. Then R is Σ^1_1.*

Proof. Recall that in the proof of Theorem 7.3, the sets T_w were defined also for all $w \in \omega^{<\omega}$ and keep in mind that for $\eta \in \mathbf{B}$, $T_\eta = \bigcup_{n<\omega} T_{\eta \restriction n}$.

Let u_i, $i < \omega$, be an enumeration of $\omega^{<\omega}$ such that if $u_i \subseteq u_j$, then $i < j$ (exercise: show that such enumeration exists). Notice that then $dom(u_i) \leq i$.

In the definition of Suslin operation, the sets X_η were needed for every $\eta \in \omega^{<\omega}$. In this proof, we use $(\omega^3)^{<\omega}$ in place of $\omega^{<\omega}$. A simple coding shows that this is harmless. Also for $w : n \to \omega^3$, we write w_i, $i < 3$, for those functions with domain n that satisfy: for all $m < n$, $w(m) = (w_i(m))_{i<3}$.

For all $w \in (\omega^3)^{<\omega}$ we let $X_w = \emptyset$ if (1), (2) or (3) below holds and otherwise, $X_w = N_{(w_0,w_1)}$.

(1) For some $k < n$, $dom(u_{w_2(k)}) \neq dom(u_k)$.

(2) For some $k < m < n$, $u_k \subsetneq u_m$ but $u_{w_2(k)} \not\subseteq u_{w_2(m)}$ or $w_2(k) = w_2(m)$.

(3) For some $m < n$, $u_m \in T_{w_0 \restriction dom(u_m)}$ but $u_{w_2(m)} \notin T_{w_1 \restriction dom(u_m)}$.

Then $R = \mathcal{A}\{X_w |\ w \in (\omega^3)^{<\omega}\}$: If $f : \omega \to \omega^3$ witnesses that $(\eta, \xi) \in \mathcal{A}\{X_w |\ w \in (\omega^3)^{<\omega}\}$, then $\cup_{n<\omega}(f \restriction n)_2$ codes an order-preserving map which witnesses that $T_\eta \leq T_\xi$. On the other hand, if $g : T_\eta \to T_\xi$ witnesses that $T_\eta \leq T_\xi$, then we find a witness $f : \omega \to \omega^3$ for $(\eta, \xi) \in \mathcal{A}\{X_w |\ w \in (\omega^3)^{<\omega}\}$ as follows: Clearly we can choose g so that $dom(g(u)) = dom(u)$ for all $u \in T_\eta$. But then we can find $g' : \omega^{<\omega} \to \omega^{<\omega}$ so that $g' \restriction T_\eta = g$ and for all $u, w \in \omega^{<\omega}$, $dom(g'(u)) = dom(u)$ and if $u \subseteq w$, then $g'(u) \subseteq g'(w)$. Let $h : \omega \to \omega$ be such that for all $n < \omega$, $g'(u_n) = u_{h(n)}$. But then $f(n) = (\eta(n), \xi(n), h(n))$ is the witness.

Thus by Lemma 8.3, R is Σ^1_1. □

Σ^1_1 sets can also be defined in a Borel* style:

8.7 Exercise. *We say that (T, π) is a Vaught code if T is the tree $(\omega^2)^{<\omega}$ and $\pi : T \to P(\mathbf{B})$. We define the Vaught game $GV(\eta, (T, \pi))$ exactly as GB^* was defined except that now the game lasts ω moves and at the end II wins if $\eta \in \pi(f \restriction n)$ for all $n < \omega$, where $f : \omega \to \omega^2$ is such that if on move n, I chose i and II chose j, then $f(n) = (i, j)$. We say that (T, π) is a Vaught code for $X \subseteq \mathbf{B}$, if for all $\eta \in \mathbf{B}$, $\eta \in X$ iff II has a winning strategy in the game $GV(\eta, (T, \pi))$. Show that*

(i) Every Σ^1_1 set X has a Vaught code (T, π) such that for all $t \in T$, $\pi(t)$ is a basic open set.

(ii) If a Vaught code (T, π) is such that for all $t \in T$, $\pi(t)$ is Σ^1_1, then the set whose Vaught code (T, π) is, is Σ^1_1.

8.8 Exercise. *Show that if every $X_\eta \subseteq \mathbf{B}$, $\eta \in \omega^{<\omega}$, is Π^1_2, then so is $X = \mathcal{A}\{X_\eta |\ \eta \in \omega^{<\omega}\}$. Hint: Choose Σ^1_1 sets $X^*_\eta \subseteq \mathbf{B}^2$ so that $\xi \in X_\eta$ iff for all $h \in \mathbf{B}$, $(\xi, h) \in X^*_\eta$. Then show that $\xi \notin X$ iff there is $h : \omega \times \omega \to \omega$ for* *which the following holds: for all $f \in \mathbf{B}$, there exists $n < \omega$ and $m < \omega$ such that $(\xi, h_m) \notin X^*_{f \restriction n}$, where $h_m : \omega \to \omega$ is such that $h_m(x) = h(m, x)$.*

9. Cardinality of Π^1_1 sets

In this section we show that if the cardinality of a Π^1_1 set is $> \omega_1$, then the cardinality is 2^ω.

Let $X \subseteq \mathbf{B}$ be Π^1_1 and choose closed $X^* \subseteq \mathbf{B} \times \mathbf{B}$ so that $\mathbf{B} - X = pr(X^*)$. And again for all $\eta \in \mathbf{B}$, let X^*_η be the set of those $\xi \in \omega^n$, $n < \omega$, such that for some $\eta', \xi' \in \mathbf{B}$, $(\eta', \xi') \in X^*$, $\eta \restriction n \subseteq \eta'$ and $\xi \subseteq \xi'$. Again we think X^*_η as a tree and notice that $\eta \in X$ iff X^*_η does not contain an ω-branch.

For all $\alpha < \omega_1$, let $X_\alpha = \{\eta \in \mathbf{B} |\ rk(X^*_\eta) < \alpha\}$. (for rk, see Definition 4.5). Notice that by Exercise 4.6 (i), $X = \bigcup_{\alpha<\omega_1} X_\alpha$.

For every ordinal α, let T_α be the tree of all $f : n \to \alpha$, $n < \omega$, such that for all $i < j < n$, $f(j) < f(i)$.

9.1 Exercise.
(i) Show that $rk(T_\alpha) = \alpha$.
(ii) Show that $rk(X^*_\eta) \geq \alpha$ iff $T_\alpha \leq X^*_\eta$.
(iii) Show that for every $\alpha < \omega_1$, X_α is Borel. Hint: See the proof of Theorem 7.3.

So we have proved:

9.2 Lemma. *Every Π^1_1 set is a union of ω_1 many Borel sets.* □

9.3 Theorem. *If $X \subseteq \mathbf{B}$ is Π^1_1 and not countable, then $|X| \in \{\omega_1, 2^\omega\}$.*

Proof. We suppose that the cardinality of X is $> \omega_1$ and show that it is 2^ω. For this it is enough to show that $|X| \geq 2^\omega$.

By Lemma 9.2, let X_α, $\alpha < \omega_1$ be such Borel sets that $X = \bigcup_{\alpha<\omega_1} X_\alpha$. By Exercise 1.4.17 (ii), there must be $\alpha < \omega_1$ such that $|X_\alpha| > \omega$. But then by Theorem 4.8, $|X_\alpha| = 2^\omega$. Since $X_\alpha \subseteq X$, $|X| \geq 2^\omega$. □

9.4 Lemma. *Every Π^1_1 set X is an intersection of ω_1 many Borel sets.*

Proof. Let $Y^* \subseteq \mathbf{B}^2$ be closed such that $pr(Y^*)$ is the complement Y of X. For all $\eta \in \mathbf{B}$, let Y_η be as before and by Y^u_η, $u \in \omega^{<\omega}$, we mean the set of those $\xi \in Y_\eta$ such that $u \subseteq \xi$. For $\alpha < \omega_1$ and $u \in \omega^{<\omega}$, by B^α_u we mean the set of those $\eta \in \mathbf{B}$ such that $rk(Y^u_\eta) = \alpha$. Then B^α_u is Borel (as in Exercise 9.1 (iii)) and thus also $E^\alpha = (\bigcup_{\beta\leq\alpha} B^\alpha_\emptyset) \cup (\bigcup_{u\in\omega^{<\omega}} B^\alpha_u)$ is Borel. Thus it suffices to show that $X = \bigcap_{\alpha<\omega_1} E^\alpha$.

Suppose first that $\eta \in X$. Then $rk(Y_\eta) < \omega_1$ and thus for all $\alpha < \omega_1$, either $rk(Y_\eta) \leq \alpha$ or for some $u \in \omega^{<\omega}$, $rk(Y^u_\eta) = \alpha$. In both cases $\eta \in E^\alpha$.

Suppose then that $\eta \notin X$ and for a contradiction, suppose that $\eta \in E^\alpha$ for all $\alpha < \omega_1$. Since $rk(Y_\eta) = \infty$, for all $\alpha < \omega_1$, there is u_α such that $rk(Y^{u_\alpha}_\eta) = \alpha$. This gives a one-to-one function from ω_1 to $\omega^{<\omega}$, a contradiction. □

9.5 Exercise. *Show that every Σ^1_2 set is a union of ω_1 many Borel sets. Conclude that uncountable Σ^1_2 sets have cardinality ω_1 or 2^ω.*

10. Uniformization

Suppose that $R \subseteq \mathbf{B}\times\mathbf{B}$. Then by Choice, there is a function $f : pr(R) \to \mathbf{B}$ such that for all $\eta \in pr(R)$, $(\eta, f(\eta)) \in R$. In uniformization questions one asks that if R is simple, can one choose f so that it(s graph i.e. $\{(\eta, f(\eta))|\ \eta \in pr(R)\}$) is also simple.

In the low end of topological complexity, without additional assumption, the answer is no:

10.1 Fact. *There is closed $R \subseteq \mathbf{B} \times \mathbf{B}$ such that if $f : pr(R) \to \mathbf{B}$ is such that for all $\eta \in pr(R)$, $(\eta, f(\eta)) \in R$, then $\{(\eta, f(\eta))|\ \eta \in pr(R)\}$) is not Σ^1_1.*

However, for Π^1_1 things are different.

10.2 Definition. *Suppose that S is a topological space and $R \subseteq S \times S$. We say that $R^* \subseteq S \times S$ uniformizes R if $R^* \subseteq R$ and for all $\eta \in pr(R) = \{\eta \in S|$ for some $\xi \in S$, $(\eta, \xi) \in R\}$, there is unique $\xi \in \mathbf{B}$ such that $(\eta, \xi) \in R^*$.*

In the proof of the next theorem, one further observation on trees is needed (this is a variant and a strengthening of what was done in Exercise 7.5). Let $T \subseteq X^{<\omega}$ be a tree. By $\sigma(T)$ we mean the tree $U \subseteq T^{<\omega}$ such that $t : n \to T$ is in $\sigma(T)$ if for all $i < j < n$, $t(i) \subsetneq t(j)$.

10.3 Exercise. *Let T and T' be trees.*
(i) Show that $rk(\sigma(T)) = rk(T) + 1$ (where $\infty + 1 = \infty$ and $-1 + 1 = 0$).
(ii) Show that $T' \not\leq T$ iff $\sigma(T) \leq T'$.
(iii) There are no trees $T_i \subseteq \omega^{<\omega}$, $i < \omega$, such that for all $i < \omega$, $T_i \not\leq T_{i+1}$.

10.4 Theorem. *If $R \subseteq \mathbf{B} \times \mathbf{B}$ is Π^1_1, then there is a Π^1_1 set $R^* \subseteq \mathbf{B} \times \mathbf{B}$ such that it uniformizes R.*

Proof. Let $T \subseteq \mathbf{B}^3$ be closed and such that $(\eta, \xi) \in R$ iff there is no $\xi' \in \mathbf{B}$ such that $(\eta, \xi, \xi') \in T$. As in the proof of Theorem 7.3, for $\eta, \xi \in \mathbf{B}$ let $T_{\eta,\xi}$ be the tree of those $f \in \omega^{<\omega}$ such that letting $n = dom(f)$, for some η', ξ' and f' from $\mathbf{B}$, $(\eta', \xi', f') \in T$, $\eta \restriction n \subseteq \eta'$, $\xi \restriction n \subseteq \xi'$ and $f \subseteq f'$. Again $(\eta, \xi) \in R$ iff $T_{\eta,\xi}$ does not contain an ω-branch.

The idea in this proof is to define R^* so that for every $\eta \in pr(R)$, R^* pics ξ so that $T_{\eta,\xi}$ is $\leq$-minimal (notice that by Exercise 7.2 (iii), this is possible) and among those, ξ is chosen so that for all n, $\xi(n)$ is the least possible. However, to make this really work, some polishing is needed.

As in the proof of Lemma 8.6, let u_i, $i < \omega$, be an enumeration of $\omega^{<\omega}$ such that if $u_i \subseteq u_j$, then $i < j$. By $T^i_{\eta,\xi}$ we mean the tree of those $t \in \omega^{<\omega}$ such that $t^i \in T_{\eta,\xi}$, where $t^i \in \omega^{<\omega}$ is such that $dom(t^i) = dom(u_i) + dom(t)$, for all $k < m = dom(u_i)$, $t^i(k) = u_i(k)$, and for $k < dom(t)$, $t^i(m + k) = t(k)$. Notice that $T^0_{\eta,\xi} = T_{\eta,\xi}$. Notice also that $rk(T^i_{\eta,\xi}) = rk(u_i; T_{\eta,\xi})$.

Now we define a set $S \subseteq \mathbf{B}^3$ so that we can choose $R^* = R \cap (\mathbf{B}^2 - S^*)$ where S^* is the set of those (η, ξ) such that for some ξ', $(\eta, \xi, \xi') \in S$.

We let S be the set of those triples $(\eta, \xi, \xi') \in \mathbf{B}^3$ for which $(*)_n(\eta, \xi, \xi')$ or $(**)_n(\eta, \xi, \xi')$ below holds for some $n < \omega$:

$(*)_n(\eta, \xi, \xi')$: $\xi' \restriction n = \xi \restriction n$, for all $i \leq n$, $T^i_{\eta,\xi'} \leq T^i_{\eta,\xi}$ and $T^n_{\eta,\xi} \not\leq T^n_{\eta,\xi'}$,

$(**)_n(\eta,\xi,\xi')$: $\xi'\restriction n=\xi\restriction n$, $\xi'(n)<\xi(n)$ and for all $i\le n$, $T^i_{\eta,\xi'}\le T^i_{\eta,\xi}$.

Recall that by Exercise 10.3 (ii), $T^i_{\eta,\xi}\not\le T^i_{\eta,\xi'}$ is equivalent with $\sigma(T^i_{\eta,\xi'})\le T^i_{\eta,\xi}$. Then $R^*=R\cap(\mathbf{B}^2-S^*)$ is Π^1_1, since

10.4.1 Exercise. *Show that S is Σ^1_1. Hint: Lemma 8.6.*

We are left to prove that R^* uniformizes R. We show first that if $\eta\in pr(R)$, then there is $\xi\in\mathbf{B}$ such that $(\eta,\xi)\in R\cap(\mathbf{B}^2-S^*)$. For all $n<\omega$, by induction, we determine $\xi(n)$ by choosing first $\xi_n\in\mathbf{B}$ and then letting $\xi(n)=\xi_n(n)$. We do this as follows (these ξ_n exist, because we minimalize in a well-founded partial order by Exercise 10.3 (iii)):
$n=0$: We let ξ_0 be any element of $\mathbf{B}$ such that for all ξ', neither $(*)_0(\eta,\xi_0,\xi')$ nor $(**)_0(\eta,\xi_0,\xi')$ hold.
$n=m+1$: We let ξ_n be any element of $\mathbf{B}$ such that $\xi_n\restriction n=\xi_m\restriction n$ and neither $(*)_n(\eta,\xi_n,\xi')$ nor $(**)_n(\eta,\xi_n,\xi')$ hold for any ξ'.
Finally we let $\xi=lim_{n\to\infty}\xi_n=\bigcup_{n<\omega}\xi\restriction(n+1)$.

Now we need to show that $(\eta,\xi)\in R$ and $(\eta,\xi)\in\mathbf{B}^2-S^*$. The proofs of these are based on the same observation.

Let us define $f:\omega^{<\omega}\to\omega_1\cup\{-1\}$ so that $f(u_n)=rk(T^n_{\eta,\xi_n})$. Recall that the rank was defined so that if $u_n\notin T_{\eta,\xi_n}$, then $f(u_n)=-1$.

10.4.2 Exercise.

(i) For all $n<m<\omega$, if $u_n=(u_m)^-$, then $f(u_m)<f(u_m)$ or $f(u_n)=f(u_m)=-1$.

(ii) If $f(u)=-1$, $u\notin T_{\eta,\xi}$.

(iii) For all $n<\omega$, $rk(u_n;T_{\eta,\xi})\le rk(u_n;T_{\eta,\xi_n})$ (in fact, equality holds here but we do not need that information). Hint: By induction on the ordering $u<w$ iff $w\subseteq u$ show that for all $u\in\{w\in\omega^{<\omega}\mid f(w)\ge 0\}$, $rk(u;T_{\eta,\xi})\le f(u)$.

(iv) $(\eta,\xi)\in R$.

(v) $(\eta,\xi)\notin S^*$. Hint: Suppose first that there are ξ' and $n<\omega$ such that $(*)_n(\eta,\xi,\xi')$ holds. Choose these so that n is minimal. Then notice that for all $i\le n$, $T^i_{\eta,\xi'}\le T^i_{\eta,\xi}\le T^i_{\eta,\xi_n}$ and $T^n_{\eta,\xi_n}\not\le T^n_{\eta,\xi'}$.

So we are left to prove that if $(\eta,\xi)\in R^*$ and $(\eta,\xi')\in R^*$, then $\xi=\xi'$. Suppose not. Let $n<\omega$ be the least such that $\xi(n)\ne\xi'(n)$. Without loss of generality, we may assume that $\xi'(n)<\xi(n)$. Since $(\eta,\xi,\xi')\notin S$ and $(\eta,\xi',\xi)\notin S$, for all $m\le n$, both $(*)_m(\eta,\xi,\xi')$ and $(*)_m(\eta,\xi',\xi)$ fail. It follows that

(***) for all $m\le n$, $T^m_{\eta,\xi'}\le T^m_{\eta,\xi}$ and $T^m_{\eta,\xi}\le T^m_{\eta,\xi'}$.

But for the same reason also $(**)_n(\eta,\xi,\xi')$ fails and thus using (***), $\xi'(n)$ can not be strictly smaller than $\xi(n)$, a contradiction. □

10.5 Exercise. *Show that if $R\subseteq\mathbf{R}^2$ is Π^1_1, then there is a Π^1_1 set $R^*\subseteq\mathbf{R}^2$ such that it uniformizes R. Hint: Use Lemma 3.13 (for $n=1$) and Theorem 10.4.*

10.6 Exercise. *Show that if $R\subseteq\mathbf{B}\times\mathbf{B}$ is Σ^1_2, then there is a Σ^1_2 set $R^*\subseteq\mathbf{B}\times\mathbf{B}$ such that it uniformizes R.*

10.7 Exercise. *Suppose that $R\subseteq\mathbf{B}\times\mathbf{B}$ is Borel and that for all $\eta\in\mathbf{B}$ there are exactly two elements $\xi\in\mathbf{B}$ such that $(\eta,\xi)\in R$. Show that there is a Borel set $R^*\subseteq\mathbf{B}\times\mathbf{B}$ such that it uniformizes R.*

11. Measurability

In this section we show that Σ^1_1 (and thus Π^1_1) sets are measurable. We will prove the result for the Lebesgue measure on $\mathbf{R}$ but from the proof it is clear that the result holds also for many other measures. For more on this, see the next section.

For $X \subseteq \mathbf{R}$, the outer measure $\mu^*(X) \in \mathbf{R}_{\geq 0} \cup \{\infty\}$ of X is the infimum of the sums $\Sigma_{n=0}^{\infty} v(I_n)$ such that each I_n is a closed interval with rational endpoints, $v(I_n)$ is the length of the interval and $X \subseteq \bigcup_{n<\omega} I_n$. A set X is Lebesgue measurable, if for all $Y \subseteq \mathbf{R}$, $\mu^*(Y \cap X) + \mu^*(Y - X) = \mu^*(Y)$. If X is Lebesgue measurable, we write $\mu(X)$ for $\mu^*(X)$ and call it the Lebesgue measure of X. A set X is null if $\mu^*(X) = 0$.

For $\mathbf{R}^n$, $n > 1$, Lebesgue measure is defined exactly the same way, only that instead of closed intervals we look at n-products of closed intervals with rational endpoints and $v(I_1 \times ... \times I_n)$ is the product of the lengths of the the closed intervals I_i, $1 \leq i \leq n$.

When there is no risk of confusion, we call Lebesgue measure just measure.

11.1 Exercise.

(i) Show that all closed intervals and null sets are measurable.

(ii) Show that if I is a closed interval, then $\mu(I) = v(I)$.

(iii) Let N be the family of all null subsets of $\mathbf{B}$. Show that N is a σ-ideal (see, Exercise 3.7 (ii)).

(iv) Show that measurable sets form a σ-algebra and so by (i), $Borel(\mathbf{R})$ sets are measurable. (recall: A family is a σ-algebra if it is closed under complements, countable unions and countable intersections).

(v) Show that for every $X \subseteq \mathbf{R}$, X is measurable iff $X \cap \mathbf{Ir}$ is measurable.

(vi) Show that if X_i, $i < \omega$, are disjoint and measurable, then $\mu(\bigcup_{i<\omega} X_i) = \Sigma_{i<\omega}\mu(X_i)$. Conclude that if X_i, $i < \omega$, are measurable and form an increasing sequence, then $\mu(\bigcup_{i<\omega} X_i) = sup_{i<\omega}\mu(X_i)$.

11.2 Lemma. *There exists a set $X \subseteq [0,1]$ such that it is not measurable.*

Proof. By Exercise 11.1, it is enough to find $X \subseteq [0,1]$ such that $\mu^*(X)$, $\mu^*([0,1] - X) \geq 2/3$ (since then $\mu^*(X) + \mu^*([0,1] - X) \geq 4/3 > 1 = \mu([0,1])$). For this we choose closed intervals $I_{i,j} \subseteq [0,1]$ with rational endpoints, $i < 2^\omega$ and $j < \omega$, so that for all $i < 2^\omega$, $\sum_{j=0}^{\infty} v(I_{i,j}) \leq 2/3$ and if $J_j \subseteq [0,1]$, $j < \omega$, are closed intervals with rational endpoints and $\sum_{j=0}^{\infty} v(J_j) \leq 2/3$, then for some $i < 2^\omega$, $J_j = I_{i,j}$ for all $j < \omega$. (This is possible since the number of closed interval with rational endpoints is countable and so the number of possible sequences $(J_i)_{i<\omega}$ is $|\omega^\omega| = 2^\omega$.) Then it is enough to find $X \subseteq [0,1]$ such that for all $i < 2^\omega$, $X \not\subseteq \bigcup_{i<\omega} I_{i,j}$ and $[0,1] - X \not\subseteq \bigcup_{i<\omega} I_{i,j}$. This is easy:

For all $i < 2^\omega$, we choose sets $X_i, Y_i \subseteq [0,1]$ so that

(i) $X_0 = Y_0 = \emptyset$,

(ii) for all $i < 2^\omega$, $X_{i+1} = X_i \cup \{x_i\}$ and $Y_{i+1} = Y_i \cup \{y_i\}$ where $x_i \in [0,1]$ and $y_i \in [0,1]$ are such that they do not belong to $\bigcup_{j<\omega} I_{i,j}$,

(iii) for all $i < 2^\omega$, $X_i \cap Y_i = \emptyset$,

(iv) for limit $i < 2^\omega$, $X_i = \bigcup_{j<i} X_j$ and $Y_i = \bigcup_{j<i} Y_j$.

To see that the sets X_i and Y_i exists, it is enough to show that for all $i < 2^\omega$, the elements x_i and y_i can be found so that the requirement in (ii) holds and $X_{i+1} \cap Y_{i+1} = \emptyset$. We start by noticing that for all $i < 2^\omega$, the cardinality of X_i and Y_i is $< 2^\omega$ (exercise). Since for all $i < 2^\omega$, the set $Z = [0,1] - \bigcup_{j<\omega} I_{i,j}$ is $Borel(\mathbf{R})$, and not null (and thus not countable), by Theorem 4.8 (and Exercise 3.4 (vii)), the cardinality of it is 2^ω. Thus for all $i < 2^\omega$, the set $Z - (X_i \cup Y_i)$ contains more that one element. So x_i and y_i can indeed be found.

Now $\bigcup_{i<2^\omega} X_i$ is as wanted. ▫

11.3 Lemma. *For every $X \subseteq \mathbf{R}$ there is a $Borel(\mathbf{R})$ (and thus measurable) set Y such that $X \subseteq Y$ and every measurable $Z \subseteq Y - X$ is null. Thus if in addition X is measurable, there is a $Borel(\mathbf{R})$ set (in fact $\Sigma_3(\mathbf{R})$ set, see the proof) Y such that $X \subseteq Y$ and $Y - X$ is null.*

Proof. By Exercise 11.1, it is easy to see that it is enough to prove the claim under the additional assumption that X is bounded (exercise). So we suppose that $X \subseteq [-r, r]$ for some $r \in \mathbf{R}_{>0}$. But then by the definition of μ^* one can find countable unions U_n of closed intervals $\subseteq [-r, r]$, $0 < n < \omega$, such that for all $0 < n < \omega$, $X \subseteq U_n$ and $\mu(U_n) - \mu^*(X) < 1/n$. Then $Y = \bigcap_{0<n<\omega} U_n$ is as wanted: Clearly, it is $Borel(\mathbf{R})$. Also if $Z \subseteq Y - X$ is measurable, then $Y - Z$ is measurable and since it contains X, $\mu(Y - Z) \geq \mu^*(X) = \mu(Y)$. So $\mu(Z) = 0$. ▫

11.4 Theorem. *Every $\Sigma^1_1(\mathbf{R}^n)$ set is measurable.*

Proof. We prove the claim for $n = 1$, the other cases are similar. Let $X \subseteq \mathbf{R}$ be $\Sigma^1_1(\mathbf{R})$ set. Since it is enough to show that $X \cap \mathbf{Ir}$ is measurable, we may assume that $X \subseteq \mathbf{Ir}$. Then, as in the proof of Lemma 8.4, choose continuous $F : \mathbf{B} \to \mathbf{Ir}$ so that $X = F(\mathbf{B})$ and for all $\eta \in \omega^{<\omega}$, let $X_\eta = F[N_\eta]$. Then (by the proof of Lemma 8.4)

(*) $X = \mathcal{A}\{X_\eta |\ \eta \in \omega^{<\omega}\} = \mathcal{A}\{\overline{X}_\eta |\ \eta \in \omega^{<\omega}\}$.

Notice also that

(**) for all $\eta \in \omega^n$, $X_\eta = \cup\{X_\xi |\ \xi \in \omega^{n+1},\ \eta \subseteq \xi\}$.

By Lemma 11.3 and since the sets $\overline{X}_\eta$ are measurable, for every $\eta \in \omega^{<\omega}$, there is a measurable set Y_η such that $X_\eta \subseteq Y_\eta \subseteq \overline{X}_\eta$ and every measurable $Z \subseteq Y_\eta - X_\eta$ is null. By (*),

(***) $X = \mathcal{A}\{Y_\eta |\ \eta \in \omega^{<\omega}\}$.

Since $Y_\emptyset$ is measurable, it is enough to show that $Y_\emptyset - X$ is measurable. For this it is enough to show that it is null.

11.4.1 Claim. *For all $n < \omega$ and $\eta \in \omega^n$, let $Z_\eta = \cup\{Y_\xi |\ \xi \in \omega^{n+1},\ \eta \subseteq \xi\}$. Then $Y_\emptyset - X \subseteq \bigcup_{n<\omega} \bigcup_{\eta \in \omega^n} (Y_\eta - Z_\eta)$.*

Proof. So suppose $a \in Y_\emptyset$ but $a \notin \bigcup_{n<\omega} \bigcup_{\eta \in \omega^n} (Y_\eta - Z_\eta)$. We need to show that $a \in X$. For this, by (***), it is enough to find $f \in \omega^\omega$ such that for all $n < \omega$, $a \in Y_{f \restriction n}$. We choose the values $f(n)$ by induction on $n < \omega$ as follows: Since $a \in Y_\emptyset$ but $a \notin Y_\emptyset - Z_\emptyset$, there is $\xi \in \omega^1$ such that $a \in Y_\xi$. Let $f(0) = \xi(0)$ (i.e. $f \restriction 1 = \xi$). Then we just keep on repeating this argument: Since $a \in Y_{f\restriction 1}$ but $a \notin Y_{f\restriction 1} - Z_{f\restriction 1}$, there is $\xi' \in \omega^2$ such that $a \in Y_{\xi'}$ and $f \restriction 1 \subseteq \xi'$. Let $f(1) = \xi'(1)$ (i.e. $f \restriction 2 = \xi'$) etc. ▫

Thus to show that $Y_\emptyset - X$ is null, it is enough to show that for all $n < \omega$ and $\eta \in \omega^n$, the set $Y_\eta - Z_\eta$ is null. But by (**), $Y_\eta - Z_\eta \subseteq Y_\eta - X_\eta$ and since $Y_\eta - Z_\eta$ is measurable, it null by the choice of Y_η. □

11.5 Fact.

(i) ZFC does not prove that $\Delta^1_2(\mathbf{R})$ sets are measurable (again assuming ZFC is consistent).

(ii) If there are infinitely many Woodin cardinals, then every projective set is measurable (see Section 14).

(iii) Under some large cardinal assumptions, ZF+"every subset of reals is measurable" is consistent.

11.6 Exercise.

(i) Show that there is $X \subseteq [0,1]$ such that $\mu^*(X) = \mu^*([0,1] - X) = 1$.

(ii) Suppose $f : \mathbf{R} \to \mathbf{R}$ is measurable i.e. for all open sets $U \subseteq \mathbf{R}$, $f^{-1}(U)$ is measurable. Show that there exists a $Borel(\mathbf{R})$ set X and a Borel function $g : \mathbf{R} \to \mathbf{R}$ such that $\mathbf{R} - X$ is null and $f \restriction X = g \restriction X$. Hint: See the proof of Corollary 3.13 and use Lemma 11.3.

11.7 Exercise. *Let $V = \{(x,y) \in \mathbf{R}^2 |\ 0 < x, y < 1\}$. Suppose $X \subseteq V$, $a \in X$ and $b \in \mathbf{R}^n$. We say that a is visible from b if the line segment (a,b) does not contain points from X. By X_b we denote the set of all points of X that are visible from b. Show that there is $X \subseteq V$ such that for all $b = (x,y) \in \mathbf{R}^2$, if $y > 2$, then $\mu^*(X_b) = 1$ (in particular, X_b has Hausdorff dimension 2). Hint: Use the idea from the proof of Lemma 11.2, Theorem A.1 from Appendix and a corollary of Fubini's theorem: If $Y \subseteq V$ is measurable and not null, then there is $0 < y < 1$ such that $\mu(\{x \in \mathbf{R} |\ (x,y) \in Y\}) > 0$.*

12. Universal measurability

As pointed out in the previous section, the proof of Theorem 11.4 goes through for many measures other than the Lebesgue measure. However, behind this there is a more general fact: measurability is a very robust notion, natural notions of measure tend to give roughly the same notion of measurability. In this section we look at this phenomenon. For simplicity we restrict to continuous Borel measures on $\mathbf{R}$ such that all bounded intervals get a finite measure.

12.1 Definition. *A function $\mu^* : Borel(\mathbf{R}) \to [0,\infty]$ is called a Borel measure on $\mathbf{R}$ if $\mu^*(\emptyset) = 0$ and μ^* is σ-additive i.e. $\mu^*(\bigcup_{i<\omega} X_i) = \Sigma_{i<\omega}\mu^*(X_i)$ for any pairwise disjoint members X_i of $Borel(\mathbf{R})$. If in addition all singletons get measure 0, we say that μ^* is continuous.*

12.2 Definition. *Let μ^* be a Borel measure on $\mathbf{R}$. We say that $X \subseteq \mathbf{R}$ is μ^*-null if there is a $Borel(\mathbf{R})$ set Y such that $X \subseteq Y$ and $\mu^*(Y) = 0$. We say that $X \subseteq \mathbf{R}$ is μ^*-measurable if there is a μ^*-null set Y such that $X \cup Y \in Borel(\mathbf{R})$.*

Let μ be the Lebesgue measure from the previous section. Then from μ, we get a continuous Borel measure $\mu \restriction Borel(\mathbf{R})$ which we also denote by μ.

12.3 Exercise.

(i) Show that $X \subseteq \mathbf{R}$ is null (in the sense of Section 11) iff X is μ-null.

(ii) Show that $X \subseteq \mathbf{R}$ is measurable (in the sense of Section 11) iff X is μ-measurable.

12.4 Theorem. *Suppose that μ^* is a continuous Borel measure on $\mathbf{R}$ and $x,y \in \mathbf{R}$ are such that $x < y$ and $\mu^*([x,y]) < \infty$. Then there is a continuous $f : [x,y] \to [0,1]$ such that for all $X \subseteq [x,y]$, X is μ^*-measurable iff $f(X)$ is μ-measurable.*

Proof. As in Section 11, it is easy to see that it is enough to prove this claim for $[x,y] = [0,1]$. Also we may assume that $\mu^*([0,1]) = 1$ (if $\mu^*([0,1]) = 0$, there is nothing to prove, choose f to be constant 0). We define $f : [0,1] \to [0,1]$ so that $f(x) = \mu^*([0,x])$.

12.4.1 Exercise. *Show that f is increasing (not necessarily properly), continuous and onto.*

Let B be the set of those $y \in [0,1]$ such that $f^{-1}(y)$ contains more than one point. We let $Y = \bigcup_{y \in B} f^{-1}(y)$.

12.4.2 Exercise.
(i) Show that B is countable, Y is Borel and $\mu^*(Y) = 0$.
(ii) Show that $f \restriction ([0,1] - Y)$ is a homeomorphism onto $[0,1] - B$.

12.4.3 Exercise.
(i) Show that X is μ^*-measurable iff $X - Y$ is μ^*-measurable.
(ii) Show that $f(X)$ is μ-measurable iff $f(X - Y)$ is μ-measurable.

Thus it is enough to prove that $X - Y$ is μ^*-measurable iff $f(X - Y)$ is μ-measurable. We notice first that for all $x \in [0,1]$, $\mu^*([0,x]) = f(x) = \mu([0, f(x)])$. Thus for all intervals (closed or not) $I \subseteq [0,1]$, $\mu^*(I) = \mu(f(I))$ (exercise). Thus for all countable unions U of intervals, $\mu^*(U) = \mu(f(U))$ (notice that we may assume that the intervals in the union are pairwise disjoint).

12.4.5 Claim. *For all $\Sigma_n([0,1]-Y)$ sets U, $0 < n < \omega$, $\mu^*(U) = \mu(f(U))$.*

Proof. We prove this by induction on $n > 0$. From what is above the claim follows for $n = 1$. So suppose it holds for n. We prove it for $n+1$. Clearly from the induction assumption it follows that the claim holds for $\Pi_n([0,1] - Y)$ sets. Since Π_n is closed under finite unions, it is enough to show that if the claim holds for $\Pi_n([0,1]-Y)$ sets $U_i \subseteq [0,1]-Y$ and $(U_i)_{i<\omega}$ form an increasing sequence, then the claim holds for $U = \bigcup_{i<\omega} U_i$. But $\mu^*(U) = sup_{i<\omega}\mu^*(U_i) = sup_{i<\omega}\mu(f(U_i)) = \mu(f(U))$. □ Claim 12.4.5

12.4.6 Exercise.
(i) Show that for all $X \subseteq [0,1] - Y$, X is μ^*-null iff $f(X)$ is μ-null.
(ii) Show that for all $X \subseteq [0,1] - Y$, X is μ^*-measurable iff $f(X)$ is μ-measurable.

□

12.5 Corollary. *Suppose that μ^* is a continuous Borel measure on $\mathbf{R}$ such that for all $x,y \in \mathbf{R}$, $\mu^*([x,y]) < \infty$. If $X \subseteq \mathbf{R}$ is $\Sigma^1_1(\mathbf{R})$, then X is μ^*-measurable.*

Proof. Exercise. □

12.6 Exercise. *Prove Corollary 12.5.*

12.7 Exercise. *What changes one needs to make to the proof of Theorem 12.4 to prove the theorem without the assumption that μ^* is continuous?*

13. The property of Baire revisited

Often, in the context of topologically simple sets, if one can prove something on measurability (ideal of null sets) the same holds for the sets with PB (ideal of meager sets) and vice versa. And, in fact, the same proof works in both cases. (In the context of arbitrary sets this connection is broken.) In this section we give an example of this. We observe that the proof of Theorem 11.4 shows also that every Σ^1_1 (and thus Π^1_1) set has PB.

13.1 Lemma. *For every $X \subseteq \mathbf{B}$ there is set Y such that it has PB, $X \subseteq Y$ and every $Z \subseteq Y - X$ with PB is meager.*

Proof. If X is meager, there is nothing to prove (choose $Y = X$). So we may assume that X is not meager. Let X' be the set of all $\eta \in \mathbf{B}$ such that for all $n < \omega$, $X \cap N_{\eta\restriction n}$ is not meager.

13.1.1 Exercise. *Show that X' is closed and $X - X'$ is meager.*

Let $Y = X \cup X'$. Then Y is a union of a closed set and a meager set and thus it has PB. Clearly $X \subseteq Y$.

Now suppose $Z \subseteq Y - X$ has PB. For a contradiction suppose that Z is not meager. Since Z has PB, there is $\xi \in \omega^{<\omega}$ such that $N_\xi - Z$ is meager and so also $N_\xi \cap X$ is meager. Also since $N_\xi - Z$ is meager, $N_\xi \cap Z \neq \emptyset$ and so there is $\eta \in \mathbf{B}$ such that $\xi \subseteq \eta$ and for all $n < \omega$, $N_{\eta\restriction n} \cap X$ is not meager. In particular, $N_\xi \cap X$ is not meager, a contradiction. □

13.2 Theorem. *Every Σ^1_1 set has PB.*

Proof. Let $X \subseteq \mathbf{B}$ be Σ^1_1 set. Then as in the proof of Lemma 8.4, choose continuous $F : \mathbf{B} \to \mathbf{B}$ so that $X = F(\mathbf{B})$ and for all $\eta \in \omega^{<\omega}$, let $X_\eta = F[N_\eta]$. Then (by the proof of Lemma 8.4)

(*) $X = \mathcal{A}\{X_\eta \mid \eta \in \omega^{<\omega}\} = \mathcal{A}\{\overline{X}_\eta \mid \eta \in \omega^{<\omega}\}$.

Notice also that

(**) for all $\eta \in \omega^n$, $X_\eta = \cup\{X_\xi \mid \xi \in \omega^{n+1},\ \eta \subseteq \xi\}$.

By Lemma 13.1 and since the sets $\overline{X}_\eta$ have PB, for every $\eta \in \omega^{<\omega}$, there is a set Y_η such that it has PB, $X_\eta \subseteq Y_\eta \subseteq \overline{X}_\eta$ and every $Z \subseteq Y_\eta - X_\eta$ with PB is meager. By (*),

(***) $X = \mathcal{A}\{Y_\eta \mid \eta \in \omega^{<\omega}\}$.

Since $Y_\emptyset$ has PB, it is enough to show that $Y_\emptyset - X$ has PB. For this it is enough to show that it is meager.

13.2.1 Claim. *For all $n < \omega$ and $\eta \in \omega^n$, let $Z_\eta = \cup\{Y_\xi \mid \xi \in \omega^{n+1},\ \eta \subseteq \xi\})$. Then $Y_\emptyset - X \subseteq \bigcup_{n<\omega}\bigcup_{\eta\in\omega^n}(Y_\eta - Z_\eta)$.*

Proof. See the proof of Claim 11.4.1. □

Thus to show that $Y_\emptyset - X$ is meager, it is enough to show that for all $n < \omega$ and $\eta \in \omega^n$, the set $Y_\eta - Z_\eta$ is meager. But by (**), $Y_\eta - Z_\eta \subseteq Y_\eta - X_\eta$ and since $Y_\eta - Z_\eta$ has PB, it is meager by the choice of Y_η. □

13.3 Exercise. *Show that there is a null set $Z \subseteq \mathbf{R}$ such that $\mathbf{R} - Z$ is meager. Hint: Show that for all $n < \omega$, there is an open and dense subset D of $\mathbf{R}$ such that $\mu(D) < 1/n$.*

14. Determinacy

In this section we study determinacy. We start with a consequence of Σ^1_1-determinacy, namely that Σ^1_1-determinacy implies that all Σ^1_2 sets are Lebesgue measurable. And then we show that the existence of a Ramsey type cardinal implies Σ^1_1-determinacy.

14.1 Definition.

(i) Let $X \subseteq \mathbf{B}$. The game $G(X)$ is defined as follows: At each move $i < \omega$, first the player I chooses $a_i \in \omega$ and then II chooses $b_i \in \omega$. The outcome of this play is the function $\eta \in \mathbf{B}$ such that for all $i < \omega$, $\eta(2i) = a_i$ and $\eta(2i+1) = b_i$. I wins if $\eta \in X$.

(ii) The winning strategies in this game are defined as before and we say that $X \subseteq \mathbf{B}$ is determined if either one of the players has a winning strategy in the game $G(X)$.

(iii) Σ^1_1-determinacy is the assumption that every Σ^1_1 set $X \subseteq \mathbf{B}$ is determined.

(iv) Projective determinacy (PD) is the assumption that every projective $X \subseteq \mathbf{B}$ is determined.

Item (i) in the following exercise is known as Gale-Steward theorem.

14.2 Exercise.

(i) Show that every closed set is determined. Hint: Suppose that II does not have a winning strategy and show that I can play so that after each of his move, the player II still does not have a winning strategy in the rest of the play and that this is a winning strategy.

(ii) Show that Σ^1_1-determinacy implies that every Π^1_1 set is determined. Hint: Notice that N_η is homeomorphic with $\mathbf{B}$ for all $\eta \in \omega^{<\omega}$.

(iii) Show that there is a non-determined set. Hint: See the proof of Lemma 11.2.

14.3 Fact. *Every Borel set is determined.*

In the proof of the next theorem, it is convenient to work in the Cantor space. We start by defining a (Haar) measure to $\mathbf{C}$: We define an outer measure as in the case of Lebesgue measure letting the outer measure of basic open sets N^c_η be $1/(2^{dom(\eta)})$. Then measurability and measure are defined exactly as in the case of Lebesgue measure. We denote this measure also by μ, it will be clear from the context whether we mean this measure or the Lebesgue measure. By repeating proofs from the case of Lebesgue measure, one can see that μ is σ-additive, all Borel sets are measurable and that Lemma 11.3 holds also for this μ and so we may think μ also as a continuous Borel measure.

Also the following exercise is needed in the proof the next theorem.

14.4 Lemma. *Denote* $\mathbf{Ir}^* = \mathbf{Ir} \cap (0,1)$. *Then there is* $F : \mathbf{Ir}^* \to \mathbf{C}$ *such that the following holds:*

(i) F *is a homeomorphism between* $\mathbf{Ir}^*$ *and* $rng(F)$.

(ii) $\mathbf{C} - rng(F)$ *is countable.*

(iii) For all $X \subseteq \mathbf{Ir}^*$, X *is measurable iff* $F(X)$ *is measurable. (In fact, if* X *is measurable, then* $\mu(X) = \mu(F(X))$.*)*

Proof. For all $\eta \in 2^{<\omega}$, define $q_\eta \in \mathbf{Q}$ by recursion on $dom(\eta)$ as follows: If $dom(\eta) = 0$, then $q_\eta = 0$ and if $dom(\eta) = n+1$, then $q_\eta = q_{\eta \restriction n}$ if $\eta(n) = 0$ and otherwise $q_\eta = q_{\eta \restriction n} + 1/2^{n+1}$. Then for every $x \in \mathbf{Ir}^*$, there is unique $\eta \in \mathbf{C}$ such that $x = sup\{q_{\eta \restriction n} |\ n < \omega\}$. We let this η be $F(x)$.

14.4.1 Exercise.

(i) Show that the set $\{q_\eta |\ \eta \in 2^{<\omega}\}$ *is dense in* $[0,1]$.

(ii) Show that F *is one-to-one.*

(iii) Show that F *is continuous.*

(iv) Show that $F^{-1} : rng(F) \to \mathbf{Ir}^*$ *is continuous.*

(v) Show that $\mathbf{C} - rng(F)$ *is countable.*

We define a linear order $<$ to $2^{\leq\omega}$ as follows: $\eta < \xi$ if $\eta \subsetneq \xi$ or there is $n \in dom(\eta) \cap dom(\xi)$ such that $\eta \restriction n = \xi \restriction n$ and $\eta(n) < \xi(n)$.

14.4.2 Exercise. *Show that* $<$ *is a linear order and that for all* $x, y \in \mathbf{Ir}^*$, $x < y$ *iff* $F(x) < F(y)$.

By Exercise 14.4.1, for all $X \subseteq [0,1]$, X is $Borel(\mathbf{R})$ iff $F(X \cap \mathbf{Ir}^*)$ is $Borel(\mathbf{C})$ and so we can define a Borel measure μ^* to $[0,1]$ by letting $\mu^*(X) = \mu(F(X \cap \mathbf{Ir}^*))$.

14.4.3 Exercise.

(i) Show that for all $\eta, \xi \in 2^{<\omega}$, $\eta < \xi$, $\mu^*(\{x \in \mathbf{Ir}^* |\ q_\eta < x < q_\xi\}) = q_\xi - q_\eta$.

(ii) Show that for all $X \subseteq \mathbf{Ir}^*$, X *is Lebesgue measurable iff it is* μ^**-measurable. Hint: The proof of Theorem 12.4.*

(iii) Show that for all $X \subseteq \mathbf{Ir}^*$, X *is Lebesgue measurable iff* $F(X)$ *is* μ*-measurable (in* $\mathbf{C}$*).*

□

14.5 Exercise. *Show with an easy proof, that* $\mathbf{Ir}^*$ *is homeomorphic with* $\mathbf{B}$ *(i.e.* $\mathbf{Ir}$*).*

14.6 Theorem. Σ^1_1*-determinacy implies that every* Σ^1_2 *subset of* $\mathbf{R}^n$, $n < \omega$, *is Lebesgue measurable.*

Proof. We prove the claim for $n = 1$, the other cases are similar. Clearly it is enough to show that every Σ^1_2 subset X^* of $\mathbf{Ir}^*$ is Lebesgue measurable. Then there is a Π^1_1 subset $Y^* \subseteq \mathbf{Ir}^* \times \mathbf{Ir}^*$ such that $X^* = pr(Y^*)$. Let F be as in Lemma 14.4. By F we denote also the function $(x,y) \mapsto (F(x), F(y))$ from $\mathbf{Ir}^2$ to $\mathbf{C}^2$. Let $X = F(X^*)$ and $Y = F(Y^*) = \{(F(\eta), F(\xi)) |\ (\eta, \xi) \in Y\}$. By Lemma 14.4, it is enough to show that X is μ-measurable.

For all $A \subseteq \mathbf{C}$, let

$$\mu_b(A) = sup\{\mu(B) |\ B \subseteq A \ Borel\}$$

and
$$\mu^b(A) = inf\{\mu(B)|\ A \subseteq B\ Borel\}.$$

Now choose a Borel set $B \subseteq X$ such that $\mu(B) = \mu_b(X)$. Let $A = X - B$. Now if $\mu^b(A) = 0$, X is measurable since then there is a Borel set C such that $B \subseteq X \subseteq C$ and $\mu(A) = \mu(C)$. Since $F^{-1}(A)$ is still Σ^1_2, it is enough to prove that if $\mu_b(X) = 0$, then $\mu^b(X) = 0$.

For this, let us define the following game $GC(Y, \epsilon)$ for $\epsilon > 0$ (C is for covering): Let W be the set of all finite unions of basic open sets and $n \mapsto U_n$ be an enumeration of W. Then at each move i, first I chooses a pair $(a_i, b_i) \in 2 \times 2$ and then II chooses $c_i < \omega$ such that $\mu(U_{c_i}) < \epsilon/(2^{3i})$ (otherwise she loses). Let $\eta, \xi \in \mathbf{C}$ be such that for all $i < \omega$, $\eta(i) = a_i$ and $\xi(i) = b_i$. Then II wins if either $(\eta, \xi) \notin Y$ or $\eta \in \bigcup_{i<\omega} U_{c_i}$.

14.6.1 Exercise. *Show that $GC(Y, \epsilon)$ is determined. Hint: Find a Σ^1_1-set $A \subseteq \mathbf{B}$ such that I has a winning strategy in $GC(Y, \epsilon)$ iff II has it in $G(A)$ and II has a winning strategy in $GC(Y, \epsilon)$ iff I has it in $G(A)$.*

We show that II has a winning strategy in the game $GC(Y, \epsilon)$. Suppose not. Then by Exercise 14.6.1, I has a winning strategy τ. Define a function $f : \mathbf{B} \to \mathbf{C} \times \mathbf{C}$ as follows: $f(h) = (\eta, \xi)$ if for all $i < \omega$, $(\eta(i), \xi(i)) = \tau(h \restriction i)$. Clearly f is continuous and $rng(f) \subseteq Y$ and thus $A = pr(rng(f)) \subseteq X$. Since $pr(F^{-1}(rng(f))) = pr((F^{-1} \circ f)(\mathbf{B}))$, $pr(F^{-1}(rng(f)))$ is Σ^1_1 and thus by Theorem 11.4, it is Lebesgue measurable and so also $A = F(pr(F^{-1}(rng(f))))$ is μ-measurable by Lemma 14.4. So $\mathbf{C} - A$ is μ-measurable and thus by Lemma 11.3 for μ, there is a $Borel(\mathbf{C})$ set B such that $\mathbf{C} - A \subseteq B$ and $\mu(B) = \mu(A)$. So letting $D = \mathbf{C} - B$, D is $Borel(\mathbf{C})$, $D \subseteq A \subseteq X$ and $\mu(D) = \mu(A)$. So $\mu(pr(rng(f))) = \mu(D) \leq \mu_b(X) = 0$. Thus there are $c_i < \omega$, $i < \omega$, such that for all $i < \omega$, $\mu(U_{c_i}) < 1/(2^{3i})$ and $rng(f) \subseteq \bigcup_{i<\omega} U_{c_i}$ (exercise). But then II wins τ by playing c_i at each move $i < \omega$, a contradiction.

So II has a winning strategy σ. For all $n < \omega$ and $(\eta, \xi) \in 2^{n+1} \times 2^{n+1}$, let $U_{\eta,\xi} = U_c$, where $c = \sigma((\eta(0), \xi(0)), ..., (\eta(n), \xi(n)))$. Now
$$X \subseteq Z = \bigcup_{n<\omega} \bigcup_{(\eta,\xi)\in 2^{n+1}\times 2^{n+1}} U_{\eta,\xi}$$
and an easy calculation shows that $\mu(Z) \leq 8\epsilon$ and thus $\mu^b(X) \leq 8\epsilon$. Since this happens with every $\epsilon > 0$, $\mu^b(X) = 0$. □

By combining Theorem 14.6 with Fact 11.5 (i), one gets:

14.7 Fact. *ZFC does not prove Σ^1_1-determinacy.*

By taking a closer look at the proof of Theorem 14.6, one gets the item (i) below:

14.8 Fact.
(i) PD implies that all projective subsets of $\mathbf{R}^n$ are Lebesgue measurable.
(ii) Σ^1_1-determinacy implies that Σ^1_2 subsets of $\mathbf{R}^n$ have PB and PD implies that projective subsets of $\mathbf{R}^n$ have PB.
(iii) PD implies that all projective subsets of $\mathbf{R}^n$ are either countable or contain a perfect set.

Let us now turn to look when Σ^1_1-determinacy holds. For a set X and $n<\omega$, we write $[X]^n$ for the set all subsets of X of size (i.e. cardinality) n. Notice that if X is a linearly ordered set, then $[X]^n$ is essentially the same set as the set of all strictly increasing functions from n to X. And, indeed, for all subsets X of On, by $[X]^\omega$ we mean the set of all strictly increasing functions from ω to X.

For cardinals κ,λ,μ and ξ, we write $\kappa\rightarrow^\lambda(\mu)^{<\omega}_\xi$ if the following holds: If f_i, $i<\lambda$, are functions from $[\kappa]^{n_i}$, $n_i<\omega$, to ξ, then there is $X\subseteq\kappa$ of power μ such that for all $i<\lambda$, $f_i\restriction[X]^{n_i}$ is constant. E.g. by Ramsey's theorem, for all $n,m<\omega$, $\omega\rightarrow^n(\omega)^{<\omega}_m$ and e.g. so called measurable cardinals κ have the property $\kappa\rightarrow^\lambda(\kappa)^{<\omega}_\mu$ for any $\lambda,\mu<\kappa$.

14.9 Exercise. *Show that for all infinite cardinals κ, $\kappa\rightarrow^1(\omega)^\omega_2$ fails i.e. that there is $F:[\kappa]^\omega\rightarrow 2$ such that for no infinite $X\subseteq\kappa$, $F\restriction[X]^\omega$ is constant. Hint: Look at the following equivalence relation E on $[\kappa]^\omega$: gEh if $rng(g)\Delta rng(h)$ is finite. For each $g\in[\kappa]^\omega$ pick $h_g\in[\kappa]^\omega$ so that gEh_g and for all $g,g'\in[\kappa]^\omega$, gEg' iff $h_g=h_{g'}$. Now define $F(g)=0$ iff $rng(g)\Delta rng(h_g)$ is even.*

To the tree $\omega^{<\omega}$ we define the following ordering: $\eta<^*\xi$ if $\xi\subsetneq\eta$ or there is $n\in dom(\eta)\cap dom(\xi)$ such that $\eta\restriction n=\xi\restriction n$ and $\eta(n)<\xi(n)$. Notice the difference between this definition and the definition for the ordering we defined to $2^{\le\omega}$ above.

14.10 Exercise.

(i) $<^*$ is a linear ordering of $\omega^{<\omega}$.

(ii) For every $X\subseteq\omega^{<\omega}$, if X is downward closed (i.e. $\xi\subseteq\eta\in X$ implies $\xi\in X$), then the following are equivalent:

(a) there is no $\eta_i\in X$, $i<\omega$, such that for all $i<\omega$, $\eta_i\subsetneq\eta_{i+1}$,

(b) restricted to X, $<^*$ is a well-ordering of X.

(iii) Suppose $X\subseteq\omega^{<\omega}$ is finite, κ is a cardinal and $f:X\rightarrow\kappa$ is order-preserving i.e. for all $\eta,\xi\in X$, if $\eta<^*\xi$, then $f(\eta)<f(\xi)$. Then f is uniquely determined by $rng(f)$.

14.11 Theorem. *If there exists a cardinal κ such that $\kappa\rightarrow^\omega(\omega_1)^{<\omega}_\omega$, then Σ^1_1-determinacy holds.*

Proof. Let $X\subseteq\mathbf{B}$ be Σ^1_1. So there is a closed $X^*\subseteq\mathbf{B}^2$ such that $X=pr(X^*)$. For all $\eta\in\mathbf{B}$, X_η is defined as in the proof of Theorem 7.3. Thus $\eta\in X$ iff X_η contains an ω-branch, which by Exercises 14.10 (ii), 1.4.17 (iii) and 1.3.7 is the same as saying that there is no order preserving $f:X_\eta\rightarrow\kappa$.

Fix an enumeration η_i, $i<\omega$, of $\omega^{<\omega}$ so that for all $i<\omega$, $dom(\eta_i)\le i$.

Let us look at the following game $GO(X^*)$: At each move $i<\omega$, first I chooses $a_i<\omega$ and then II chooses $b_i<\omega$ and $c_i\in\kappa\cup\{-1\}$. When all ω moves are played the winner is determined as follows: Let $\eta\in\mathbf{B}$ be such that for all $i<\omega$, $\eta(2i)=a_i$ and $\eta(2i+1)=b_i$ and $f:X_\eta\rightarrow\kappa\cup\{-1\}$ be such that for all $\xi\in X_\eta$, $f(\xi)=c_i$ if $\xi=\eta_i$. Then II wins if $rng(f)\subseteq\kappa$ and f is order preserving.

14.11.1 Exercise. *Show that $GO(X^*)$ is determined i.e. one of the players has a winning strategy. Hint: See the hint for Exercise 14.2 (i).*

So it is enough to show that if I has a winning strategy in $GO(X^*)$ then he has it is $G(X)$ and if II has a winning strategy in $GO(X^*)$ then she has it is $G(X)$.

14.11.2 Exercise. *Show that if II has a winning strategy in $GO(X^*)$ then she has it in $G(X)$.*

So we are left to prove the following claim.

14.11.3 Claim. *If I has a winning strategy in $GO(X^*)$ then he has it in $G(X)$.*

Proof. Let τ be the winning strategy. For $\eta \in \omega^{2n}$, $n < \omega$, by $X(\eta)$ we denote the set of those η_i such that $i < n$ and $\eta_i \in X_\xi$, where ξ is some element of $\mathbf{B}$ such that $\eta \subseteq \xi$. Notice that by our choice of the enumeration of $\omega^{<\omega}$, $X(\eta)$ does not depend on the choice of ξ. For all $\eta \in \omega^{2n}$, $n < \omega$, let n_η be the cardinality of $X(\eta)$. Also for all $a_0, b_0, a_1, ..., b_n < \omega$, by $X(a_0, b_0, a_1, ..., b_n)$ we mean $X(\eta)$, where $\eta \in \omega^{2n+1}$ is such that for all $i \leq n$, $\eta(2i) = a_i$ and $\eta(2i+1) = b_i$.

For $n < \omega$, $a \in [\kappa]^m$, $m < \omega$, and $X \subseteq \{\eta_i |\ i < n\}$, if $m = |X|$, then by $a^*(n, X)$ we mean the function from $\{\eta_i |\ i < n\}$ to $\kappa \cup \{-1\}$ such that $a^*(n, X) \restriction X$ is order preserving and onto a and for $x \in \{\eta_i |\ i < n\} - X$, $a^*(n, X)(x) = -1$. If n and X are clear from the context, we drop them from the notation.

For all $\eta \in \omega^{2n}$, $n < \omega$, we define a coloring $F_\eta : [\kappa]^{n_\eta} \to \omega$ as follows: let $a \in [\kappa]^{n_\eta}$ and let $a^* = a^*(n, X(\eta))$. Then $F_\eta(a) = \tau((\eta(2i+1), a^*(\eta_i))_{i<n})$. Since $\kappa \to^\omega (\omega_1)^{<\omega}_\omega$, there is uncountable $Y \subseteq \kappa$ such that every F_η is constant on $[Y]^{n_\eta}$.

14.11.3.1 Subclaim. *For all $n < \omega$ and $b_0, ..., b_{n-1} < \omega$ (i.e. if $n = 0$, then there are no b_i's), there are $a_0, ..., a_n < \omega$ such that (*)$(n, b_0, ..., b_{n-1})$ holds, where*
(*)$(n, b_0, ..., b_{n-1})$: letting $\eta \in \omega^{2n}$ be such that for all $i < n$, $\eta(2i) = a_i$ and $\eta(2i+1) = b_i$, the following holds: For any $a \in [Y]^{n_\eta}$ and $i \leq n$, $a_i = \tau((b_0, a^*(\eta_0)), ..., (b_{i-1}, a^*(\eta_{i-1})))$, where $a^* = a^*(n, X(\eta))$.
Furthermore, if a'_i, $i \leq k$, are elements such that they satisfy (*)$(k, b_0, ..., b_{k-1})$, then $a'_i = a_i$ for all $i \leq n$.

Proof. We prove this by induction on n.

$n = 0$: We let $a_0 = \tau(\emptyset)$ (I chooses first). Clearly this is as wanted.

$n = k+1$: Let $a_0, ..., a_k$ be such that they satisfy (*)$(k, b_0, ..., b_{k-1})$. Let $\eta \in \omega^{2k}$ be such that for all $i < k$, $\eta(2i) = a_i$ and $\eta(2i+1) = b_i$. We notice that $dom(\eta_k) \leq 2k$ by the choice of our enumeration and so

(**) $\eta_k \in X(\eta)$ iff $\eta_n \in X(\xi)$ for some (any) $\xi \in \omega^{2n}$ such that $\eta \subseteq \xi$.

There are two cases: the case when $\eta_k \notin X(\eta)$ and the case when $\eta_k \in X(\eta)$. We leave the former case as an exercise and give the proof in the latter case.

Let $p = n_\eta + 1$ and $a \subseteq [Y]^p$. Let Let $x \in a$ be such that $|\{y \in a |\ y < x\}| = |\{\xi \in X(\eta) |\ \eta_i <^* \eta_k\}|$ and $b = a - \{x\}$. Let $a^* = a(n, X(\eta) \cup \{\eta_k\})$ and $b^* = b(k, X(\eta))$. Notice that for all $i < k$, $a^*(\eta_i) = b^*(\eta_i)$. Now we let $a_n = \tau((b_0, a^*(\eta_0)), ..., (b_k, a^*(\eta_k))$

Now by the induction assumption for every $i < n$,

$$a_i = \tau((b_0, b^*(\eta_0)), ..., (b_{i-1}, b^*(\eta_{i-1})) = \tau((b_0, a^*(\eta_0)), ..., (b_{i-1}, a^*(\eta_{i-1}))$$

and so by the induction assumption and the choice of Y, for every $i \le n$ and $a \in [Y]^p$, $a_i = \tau((b_0, a^*(\eta_0)), ..., (b_{i-1}, a^*(\eta_{i-1}))$. Finally, since by (**) $p = n_\xi$, where $\xi \in \omega^{2n}$ is such that $\eta \subseteq \xi$, $\xi(2k) = a_k$ and $\xi(2k+1) = b_k$, the elements $a_0, ..., a_n$ satisfy (*) $(n, b_0, ..., b_{n-1})$. The furthermore part follows from our choice of $a_0, ..., a_k$ (notice that if $a'_0, ..., a'_{k-1}$ satisfy (*) $(k, b_0, ..., b_{k-1})$, then for all $i < k$, $a'_i = a_i$). □ Subclaim 14.11.3.1.

Now we can define a strategy τ^* for I in the game $G(X)$. At each move n, if II has played on earlier move $b_0, ..., b_{n-1}$, then I chooses $a_0, ..., a_n$ so that they satisfy (*) $(n, b_0, ..., b_{n-1})$ and then I plays a_n.

We are left to show that τ^* is winning. For a contradiction suppose that by choosing b_i, $i < \omega$, II can beat τ^*. Let $\eta \in \mathbf{B}$ be such that for all $i < \omega$, $\eta(2i) = \tau^*(b_0, ..., b_{2i-1})$ and $\eta(2i+1) = b_i$. Then $\eta \notin X$ and thus X_η is well-ordered by $<^*$. So there is an order preserving $g : X_\eta \to Y$. But then in the game $GO(X^*)$ II can beat τ by choosing at every move $i < \omega$, b_i and $g(\eta_i)$ if $\eta_i \in X_\eta$ and if $\eta_i \notin X_\eta$, then II chooses b_i and -1. This is a win for II, because by Subclaim 14.11.3.1, for every $i < \omega$, $\tau((b_0, g(\eta_0)), ..., (b_{i-1}, g(\eta_{i-1}))) = \tau^*(b_0, ..., b_{i-1})$, a contradiction. □ Claim 14.11.3

□

14.12 Corollary. *If there exists a cardinal κ such that $\kappa \to^\omega (\omega_1)^{<\omega}_\omega$, then every Σ^1_2 (and thus Π^1_2) subset of $\mathbf{R}^n$ is Lebesgue measurable.* □

Item (i) in the following fact follows from Corollary 14.12 together with Fact 14.7.

14.13 Fact.

(i) ZFC does not prove the existence of a cardinal κ with the property $\kappa \to^\omega (\omega_1)^{<\omega}_\omega$.

(ii) If there are infinitely many Woodin cardinals, then PD holds.

15. Borel model classes and $L_{\omega_1\omega}$

In this section we look at the connection between being Borel and being definable in the infinitary language $L_{\omega_1\omega}$. We fix a language L i.e. a collection of relation, function and constant symbols. We could let L be any countable language but for simplicity we let $L = \{R\}$, where R is a binary relation symbol (i.e. just some symbol with a fancy name). An L-structure is a pair $M = (U(M), R^M)$, where $U(M)$ is a non-empty set and $R^M \subseteq U(M)^2$. In this section we look only those L-structure M in which $U(M) = \omega$. We fix a one-to-one and onto function $F : \omega^2 \to \omega$ such that

(*) for all $i, j \in \omega$, $F(i,j) \ge max\{i, j\}$.

Then every $\eta \in \mathbf{B}$ codes a model $M_\eta = (\omega, R^{M_\eta})$, where R^{M_η} is such that $(n, m) \in R^{M_\eta}$ if $\eta(F(n,m)) > 0$. Also for every $M = (\omega, R^{\mathbf{M}})$, there is $\eta \in \mathbf{B}$ such that $M = M_\eta$.

We pick some symbols v_i, $i < \omega$, and call them variables.

15.1 Definition. *The set of $L_{\omega_1\omega}$-formulas is defined by recursion as follows:*

(i) $R(v_i, v_j)$ and $v_i = v_j$ are $L_{\omega_1\omega}$-formulas (these are called atomic formulas),

(ii) if ϕ is an $L_{\omega_1\omega}$-formula, then $\neg\phi$ is an $L_{\omega_1\omega}$-formula,

(iii) if ϕ_i, $i<\omega$, are $L_{\omega_1\omega}$-formulas, then $(\bigwedge_{i<\omega}\phi_i)$ is an $L_{\omega_1\omega}$-formula,

(iv) if ϕ is an $L_{\omega_1\omega}$-formula, then $\exists v_i\phi$ is an $L_{\omega_1\omega}$-formula.

Since all the formulas we look, are $L_{\omega_1\omega}$-formulas, we call them just formulas. The following notation is used:

$$\bigvee_{i<\omega}\phi_i = \neg(\bigwedge_{i<\omega}\neg\phi_i)$$

$$\forall v_i\phi = \neg\exists v_i\neg\phi.$$

15.2 Definition. *The notion v_i is free in ϕ is defined as follows:*
(i) ϕ is atomic: v_i is free in ϕ if v_i appears in ϕ,
(ii) $\phi = \neg\psi$: v_i is free in ϕ if it is free in ψ,
(iii) $\phi = \bigwedge_{i<\omega}\psi_i$: v_i is free in ϕ if it is free in some ψ_i,
(iv) $\phi = \exists v_j\psi$: v_i is free in ϕ if it is free in ψ and $i \neq j$.

We say that a formula ϕ is a sentence if no v_i appear free in ϕ. Now we are ready to define the truth of a formula in a structure with an interpretation:

15.3 Definition. *For a formula ϕ, $s:\omega\to\omega$ and a structure M with $U(M)=\omega$, $M\models_s\phi$ is defined as follows:*
(i) $\phi = v_i = v_j$: $M\models_s\phi$ if $s(i)=s(j)$,
(ii) $\phi = R(v_i,v_j)$: $M\models_s\phi$ if $(s(i),s(j))\in R^M$,
(iii) $\phi=\neg\psi$: $M\models_s\phi$ if $M\not\models_s\psi$,
(iv) $\phi=\bigwedge_{i<\omega}\psi_i$: $M\models_s\phi$ if $M\models_s\psi_i$ for all $i<\omega$,
(v) $\phi=\exists v_i\psi$: $M\models_s\phi$ if there is $b\in\omega$ such that $\mathbf{M}\models_{s(b/i)}\psi$ where $s(b/i):\omega\to\omega$ is such that $s(b/i)(j)=b$ if $j=i$ and otherwise $s(b/i)(j)=s(j)$.

15.4 Exercise. *Show that if ϕ and $s,s':\omega\to\omega$ are such that $s(i)=s'(i)$ for all $i<\omega$ such that v_i appears free in ϕ, then $M\models_s\phi$ iff $M\models_{s'}\phi$.*

So if ϕ is a sentence, the truth does not depend on the choice of s, and thus we write just $\mathbf{M}\models\phi$ meaning that for some (every) s, $M\models_s\phi$. And if this is the case we say that $\mathbf{M}$ is a model of ϕ. Similarly, if $X\subseteq\omega$ contains all i such that v_i is free in ϕ and $u:X\to\omega$, we write $M\models_u\phi$ for some (every) $s\supseteq u$, $M\models_s\phi$.

15.5 Exercise.

(i) Let ϕ be a formula and X_ϕ be the set of those $(\eta,\xi)\in\mathbf{B}^2$ such that $M_\eta\models_\xi\phi$. Show that X_ϕ is Borel. Hint: For all $n,i<\omega$, the function $S^i_n:\mathbf{B}^2\to\mathbf{B}^2$, $S^i_n(\eta,\xi)=(\eta,\xi(n/i))$ is continuous.

(ii) Let ϕ be a sentence and $Y_\phi\subseteq\mathbf{B}$ be the set of those η such that $M_\eta\models\phi$. Show that Y_ϕ is Borel.

We will look at an alternative way of defining the semantics for sentences and an alternative way of showing that Y_ϕ is Borel.

15.6 Definition. *Semantic game $GS(M,\phi,v)$ for an L-structure $M=(\omega,R^M)$, $L_{\omega_1\omega}$-formula ϕ and $v: X\to\omega$ where X is such that it contains all i such that v_i is free in ϕ, is defined the following way: The game start at the position (II,v,ϕ). At each round n, if the position is (X,u,ψ), then the players moves as follows (by Y we denote the element $Y\in\{I,II\}-\{X\}$):*

(i) if $\psi=\neg\theta$, then the players do nothing and the game continues from the position (Y,u,θ),

(ii) if $\psi=\bigwedge_{i<\omega}\psi_i$, then Y chooses some $i<\omega$ and the game continues from the position (X,u,ψ_i),

(iii) if $\psi=\exists v_j\theta$, then X chooses $i<\omega$ and the game continues from the position $(X,u(i/j),\theta)$,

(iv) if $\psi=R(v_i,v_j)$, then the game ends and II wins if either $(u(i),u(j))\in R^M$ and $X=II$ or $(u(i),u(j))\notin R^M$ and $X=I$,

(v) if $\psi=v_i=v_j$, then II wins if either $u(i)=u(j)$ and $X=II$ or $u(i)\neq u(j)$ and $X=I$.

15.7 Exercise. *Show that $M\models_s\phi$ iff II has a winning strategy for $GS(M,\phi,s)$.*

Now fix an $L_{\omega_1\omega}$-sentence ϕ. For all $n<\omega$, we define sets $T_n\subseteq(\omega\times\omega)^n$ and a labeling L_n the following way: $T_0=\{\emptyset\}$ and $L_0(\emptyset)=(II,\emptyset,\phi)$. Then suppose that T_n and L_n are defined. Then the elements of T_{n+1} are got as follows: Suppose $\eta\in T_n$ and $L_n(\eta)=(X,u,\psi)$.

(i) If $\psi=\neg\theta$, then for all $i,j<\omega$, $\xi=\eta\cup\{(n,(i,j))\}\in T_{n+1}$ and $L_{n+1}(\xi)=(Y,u,\theta)$ where $Y\in\{I,II\}-\{X\}$,

(ii) if $\psi=\bigwedge_{i<\omega}\psi_i$, then for all $i,j<\omega$, $\xi=\eta\cup\{(n,(i,j))\}\in T_{n+1}$ and $L_{n+1}(\eta\cup\{(n,(i,j))\})=(X,u,\psi_i)$ if $X=II$ and otherwise $L_{n+1}(\eta\cup\{(n,(i,j))\})=(X,u,\psi_j)$,

(iii) if $\psi=\exists v_k\theta$, then for all $i,j<\omega$, $\xi=\eta\cup\{(n,(i,j))\}\in T_{n+1}$ and $L_{n+1}(\eta\cup\{(n,(i,j))\})=(X,u(j/k),\theta)$, if $X=II$ and otherwise $L_{n+1}(\eta\cup\{(n,(i,j))\})=(X,u(i/k),\theta)$.

If none of (i)-(iii) hold, then no $\xi\in T_{n+1}$ extend η. We let $T=\cup_{n<\omega}T_n$.

15.8 Exercise. *Show that there is no branch in T of length ω.*

Now we define a labeling π to the leafs of T. Let $\eta\in(\omega\times\omega)^n$ be a leaf of T and let $L_n(\eta)=(X,u,\psi)$. Then ψ must be atomic and thus there are two possibilities:

(i) $\psi=R(v_i,v_j)$: if $X=II$, then we let $\pi(\eta)$ be the set of all $\xi\in\mathbf{B}$ such that $\xi(F(u(i),u(j)))\geq 1$ and otherwise we let $\pi(\eta)$ be the set of all $\xi\in\mathbf{B}$ such that $\xi(F(u(i),u(j)))=0$,

(ii) $\psi=v_i=v_j$: we let $\pi(\eta)$ be $\mathbf{B}$ if either $X=II$ and $u(i)=u(j)$ or $X=I$ and $u(i)\neq u(j)$ and otherwise we let $\pi(\eta)=\emptyset$.

15.9 Exercise. *Show that (T,π) is a Borel*-code for Y_ϕ (see Exercise 15.5 (ii)).*

15.10 Definition.

(i) Suppose M and N are L-structures. We say that M and N are isomorphic, if there is a one-to-one function f from M onto N such that for all $a,b\in M$, $(a,b)\in R^M$ iff $(f(a),f(b))\in R^N$.

(ii) We say that $\eta, \xi \in \mathbf{B}$ are isomorphic if M_η and M_ξ are isomorphic.

(iii) We say that $X \subseteq \mathbf{B}$ is closed under isomorphisms if for all isomorphic $\eta, \xi \in \mathbf{B}$, $\eta \in X$ iff $\xi \in X$.

15.11 Exercise. *Let ϕ be a sentence. Show that Y_ϕ is closed under isomorphisms. (For Y_ϕ, see Exercise 15.5.)*

15.12 Theorem. *Suppose $X \subseteq \mathbf{B}$ is closed under isomorphisms. Then X is Borel iff there is an $L_{\omega_1\omega}$-sentence ϕ such that X is the set off all $\eta \in \mathbf{B}$ such that $M_\eta \models \phi$.*

Proof. From right to left, the claim is proved in Exercise 15.5 (ii). So we prove the other direction.

15.12.1 Exercise. *Show that the set S_ω of those $\eta \in \mathbf{B}$, which are permutations of ω (i.e. one-to-one and onto), is Borel but not dense.*

For all $u \in \omega^{<\omega}$, by S_u we mean the set $\{p \in S_\omega |\ u \subseteq p^{-1}\}$. Then S_u is a topological space with the induced topology. Notice that if u is not one-to-one, then $S_u = \emptyset$. Since we do not want to keep repeating all the time that u is one-to-one, we use in this proof a convention, that when we talk about elements u, v etc. from $\omega^{<\omega}$, we mean those elements of $\omega^{<\omega}$ that are one-to-one.

15.12.2 Exercise. *Show that Baire's theorem (Lemma 3.6) holds for S_u.*

So when we say that $Y \subseteq S_u$ is co-meager, we mean that it is co-meager in S_u (by Exercise 15.12.1, it can not be co-meager in any other sense).

We notice that for all $\eta \in \mathbf{B}$, there is $\xi \in \mathbf{C}$ such that M_ξ is isomorphic with M_η. Thus it is enough to show that if $A \subseteq \mathbf{C}$ is $Borel(\mathbf{C})$ and closed under isomorphisms, then there is a sentence ϕ such that for all $\eta \in \mathbf{C}$, $\eta \in A$ iff $M_\eta \models \phi$.

S_ω acts on $\mathbf{C}$ by $p\eta = \xi$, $p \in S_\omega$ and $\eta, \xi \in \mathbf{C}$, if p is an isomorphism from M_η onto M_ξ (i.e. if $F(i,j) = n$, then $\xi(n) = \eta(F(p^{-1}(i), p^{-1}(j)))$). For every $A \subseteq \mathbf{C}$ and $u \in \omega^{<\omega}$, by A^{*u} we mean the set of those $\eta \in \mathbf{C}$ such that the set $\{p \in S_u |\ p\eta \in A\}$ is co-meager. We say that A^{*u} is $L_{\omega_1\omega}$-definable if there is a formula ϕ in which only v_i, $i < dom(u)$ appear free and for all $\eta \in \mathbf{C}$, $\eta \in A^{*u}$ iff $M_\eta \models_u \phi$.

Let Z be the set of all Borel sets $A \subseteq \mathbf{C}$ such that A^{*u} is $L_{\omega_1\omega}$-definable for all $u \in \omega^{<\omega}$.

15.12.3 Exercise. *Show that it suffices to show that $Z = Borel(\mathbf{C})$.*

We prove that $Z = Borel(\mathbf{C})$ by induction i.e. that Z contains all basic open sets, is closed under complements and countable intersections. To keep the induction going, we prove a bit stronger claim: We show that for all Borel sets $A \subseteq \mathbf{C}$, the formulas ϕ^A_u that define A^{*u} can be chosen so that if $dom(u) = dom(u')$, then $\phi^A_u = \phi^A_{u'}$ i.e. that the formula depends on u only upto $dom(u)$. So we denote ϕ^A_u also by ϕ^A_n, where $n = dom(u)$.

We start by showing that every basic open set N_w, $w \in 2^{<\omega}$, is in Z: Fix some $u \in \omega^{<\omega}$. We need to show that $(N_w)^{*u}$ is $L_{\omega_1\omega}$-definable (with the extra requirement). For all $n < dom(w)$, let i, j be such that $F(i,j) = n$ and ψ_n be $R(v_i, v_j)$ if $w(n) = 1$ and otherwise $\psi_n = \neg R(v_i, v_j)$. Let I the set of all

$i < \omega$ such that for some $j < \omega$, $F(i,j) < dom(w)$ or $F(j,i) < dom(w)$. Let $J = I - dom(u)$.

15.12.4 Exercise.

(i) Show that $\eta \in (N_w)^{*u}$ iff for all distinct $a_i \in \omega - rng(u)$, $i \in J$, the following holds: If $s \in \omega^\omega$ is such that $u \subseteq s$ and for all $i \in J$, $s(i) = a_i$, then for all $n < dom(w)$, $M \models_s \psi_n$.

(ii) Show that the required ϕ_u^A exists.

Let us then look at the case when $A = \bigcap_{i<\omega} A_i$ and each $A_i \in Z$ satisfies our claim. Let $u \in \omega^{<\omega}$ be arbitrary. Clearly it suffices to show that $A^{*u} = \bigcap_{i<\omega} (A_i)^{*u}$.

By the definition of A^{*u}, for all $\eta \in 2^\omega$, $\eta \in A^{*u}$ iff

(1) $\{p \in S_u | \; p\eta \in \bigcap_{i<\omega} A_i\}$ is co-meager.

Since countable intersections of co-meager sets are co-meager and the set of co-meager sets is closed upwards, (1) is equivalents with

(2) for all $i < \omega$, $\{p \in S_u | \; p\eta \in A_i\}$ is co-meager.

By the definition of $(A_i)^{*u}$, (2) is equivalent with

(3) for all $i < \omega$, $\eta \in (A_i)^{*u}$

i.e. $\eta \in \bigcap_{i<\omega} (A_i)^{*u}$.

Finally, suppose that $A = 2^\omega - B$ and $B \in Z$ satisfies our claim. Let $u \in \omega^{<\omega}$. For all $X \subseteq 2^\omega$, we will write X^c for $2^\omega - X$. With this notation, we will prove that

(4) $A^{*u} = \bigcap_{u \subseteq v \in \omega^{<\omega}} (B^{*v})^c$.

15.12.5 Exercise. *Show that it is enough to prove (4).*

We start by noticing that for all $\eta \in 2^\omega$, the function $p \mapsto p\eta$ is a continuous function from S_u to 2^ω. Thus, since B is $Borel(\mathbf{C})$, the set $\{p \in S_u | \; p\eta \in B\}$ is $Borel(S_u)$, in particular

15.12.6 Exercise. *Show that the set $X = \{p \in S_u | \; p\eta \in B\}$ has PB in the space S_u i.e. there is an open $U \subseteq S_u$ such that $X \Delta U$ is meager in S_u.*

Suppose first that $\eta \notin A^{*u}$ i.e. the set $\{p \in S_u | \; p\eta \in B\}$ is not meager (in S_u). By Exercise 15.12.6, there is $v \in \omega^{<\omega}$ such that $S_v - \{p \in S_u | \; p\eta \in B\}$ is meager (in S_u) i.e. $\{p \in S_v | \; p\eta \in B\}$ is co-meager (in S_v). So $\eta \in B^{*v}$ and thus $\eta \notin \bigcap_{u \subseteq v \in \omega^{<\omega}} (B^{*v})^c$.

Suppose then that $\eta \in A^{*u}$ i.e. $\{p \in S_u | \; p\eta \in B\}$ is meager. But then for all $u \subseteq v \in \omega^{<\omega}$, $\{p \in S_v | \; p\eta \in B\}$ is also meager (in S_v) i.e. $\eta \notin B^{*v}$ and so $\eta \in \bigcap_{u \subseteq v \in \omega^{<\omega}} (B^{*v})^c$. □

15.13 Exercise. *Let $X \subseteq \mathbf{B}^2$ be closed and $W \subseteq \mathbf{B}$ be the set of all $\eta \in \mathbf{B}$ such that M_η is isomorphic with $(T, \subseteq)$ for some tree $T \subseteq \omega^\omega$ such that it does not contain an ω-branch. By X_η^* we mean $X_\eta \cup \omega^{\leq 1}$.*

(i) Show that there is a continuous $f : \mathbf{B} \to \mathbf{B}$ such that for all $\eta \in \mathbf{B}$, if $\xi = f(\eta)$, then M_ξ is isomorphic with $(X_\eta^*, \subseteq)$.

(ii) Show that W is not Borel and thus not $L_{\omega_1\omega}$-definable.

(iii) Why in (i) we used X_η^* instead of X_η?

(iv) Show that the set WO of those $\eta \in \mathbf{B}$ such that M_η is a well-order, is not $L_{\omega_1\omega}$-definable.

15.14 Exercise. *Show that there is $E \subseteq \mathbf{B}^2$ such that it is an equivalence relation, Σ^1_1 and the number of E-equivalence classes is ω_1. Hint: Think countable ordinals.*

15.15 Fact. *If $E \subseteq \mathbf{B}^2$ is a Σ^1_1 equivalence relation, then the number of E-equivalence classes is either $\leq \omega_1$ or 2^ω.*

A famous open question asks, is it true that the number of countable models of an $L_{\omega_1\omega}$-sentence is up to isomorphism always either $\leq \omega$ or 2^ω? By Fact 15.15, the number is always either $\leq \omega_1$ or 2^ω. If in this question one replaces $L_{\omega_1\omega}$-sentence by a countable complete first-order theory, one gets even more famous open question known as Vaught's conjecture.

Appendix: Cardinals revisited

A.1 Theorem. *For all non-empty sets a and b, if one of them is infinite, then $|a \times b| = max\{|a|, |b|\}$.*

Proof. Clearly, it is enough to prove that for all infinite cardinals κ, $|\kappa \times \kappa| = \kappa$. For this it is enough to find a one-to-one function from $\kappa \times \kappa$ to κ. We order the elements of $On \times On$ so that $(\alpha, \beta) < (\gamma, \delta)$ if one of the following holds:
(i) $max\{\alpha, \beta\} < max\{\gamma, \delta\}$,
(ii) $\alpha < \gamma \leq max\{\alpha, \beta\} = max\{\gamma, \delta\}$,
(iii) $\alpha = max\{\alpha, \beta\} = max\{\gamma, \delta\} = \gamma$ and $\beta < \delta$.

A.1.1 Exercise. *Show that $<$ is a well-ordering of $On \times On$.*

Using Theorem 1.2.3, define $\Gamma : On \times On \to On$ so that for all $x \in On \times On$, $\Gamma(x)$ is the least ordinal (strictly) greater than every element in $rng(\Gamma \restriction (On \times On)_x)$ (for this notation, see Theorem 1.2.3).

A.1.2 Exercise. *Show that F is strictly increasing and that if $\Gamma(\alpha, \beta) = \gamma$ and $\gamma' < \gamma$, then there is $(\alpha', \beta') < (\alpha, \beta)$ such that $\Gamma(\alpha', \beta') = \gamma'$.*

By Exercise A.1.2, it is enough to show that for infinite cardinals κ, $rng(\Gamma \restriction (\kappa \times \kappa)) \subseteq \kappa$. We do this by induction. The case when $\kappa = \omega$ is left as an exercise. So suppose $\kappa > \omega$. For a contradiction suppose that there are $\alpha, \beta < \kappa$ such that $\Gamma(\alpha, \beta) \geq \kappa$. Let $\lambda = max\{|\alpha|, |\beta|\} < \kappa$. Then by Exercise A.1.2, $\Gamma^{-1} \restriction \kappa : \kappa \to (On \times On)_{(\alpha,\beta)}$ is one-to-one and by the induction assumption (from which it follows that if $|a|, |b| \leq \lambda$, then $|a \times b| \leq \lambda$), $|(On \times On)_{(\alpha,\beta)}| \leq |max\{\alpha, \beta\} \times max\{\alpha, \beta\}| = |\lambda \times \lambda| = \lambda$, a contradiction. □

A.2 Exercise.
(i) Suppose κ is an infinite cardinal and a is a set of cardinality $\leq \kappa$ such that also every element of it is of cardinality $\leq \kappa$. Show that $|\cup a| \leq \kappa$. In particular, for all sets a and b, if one of them is infinite, then $|a \cup b| = max\{|a|, |b|\}$.
(ii) For all infinite cardinals κ, show that there are sets $X_i \subseteq \kappa$, $i \in \kappa$, such that for all i, the cardinality of X_i is κ and for all $i \neq j$, $X_i \cap X_j = \emptyset$.

A.3 Lemma. *For all cardinals κ, $|P(\kappa)| = |2^\kappa|$ and if κ is infinite, then $|2^\kappa| = |(2^\kappa)^\kappa| = |2^{(\kappa \times \kappa)}| = |\kappa^\kappa|$.*

Proof. As in the case $\kappa = \omega$. □

A.4 Theorem. *For all sets a, $|P(a)| > |a|$.*

Proof. Clearly it is enough to prove the claim in the cases when a is some cardinal κ, i.e. that $2^\kappa > \kappa$. For finite cardinals the claim is clear and for infinite this can be proved as in the case $\kappa = \omega$. □

A.5 Definition. *If κ is a cardinal, then the least cardinal λ greater that κ is denoted by κ^+. If κ is λ^+ for some cardinal λ, it is called a successor cardinal and otherwise it is a limit cardinal.*

A.6 Exercise.

(i) Show that for all ordinals α, there is a cardinal $\kappa > \alpha$.

(ii) Show that every infinite successor cardinal is regular.

(iii) Let X, Y, I and α_i and f_i, $i \in I$, be as in Definition 1.5.1 (ii). Suppose further that κ is a regular cardinal such that for all $i \in I$, $\alpha_i < \kappa$. Then $C(Y, f_i)_{i\in I} = C_\kappa(Y, f_i)_{i\in I}$.

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