Inverse problems course, spring 2013 Exercise 4 (22.2.2013)
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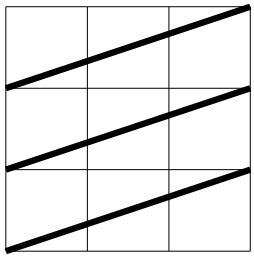
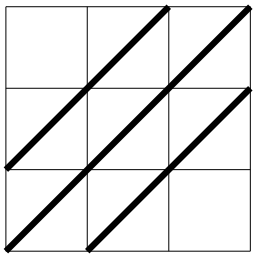
Note that this exercise has more than one page.

Please complete the theoretical exercises (marked with T) before the exercise session and be prepared to present your solution there.

T1. Recall the following model from Exercise 3. Thin lines depict pixels and thick lines X-rays in this image:





Take the same matrix A and the numbering of pixels ($f \in \mathbb{R}^9$) and X-rays ($m \in \mathbb{R}^6$) you constructed in Exercise 3.

(a) Construct a specific target $f \in \mathbb{R}^9$ of imaging by placing value one in the center pixel and value zero in other pixels. Compute measurement data $m = Af$ by matrix multiplication.

(b) Take the data m constructed in (a) and multiply it with the transpose matrix A^T. Express the result as a 3×3 image.

(c) Based on the structure of the image constructed in (b), explain why A^T is often called *backprojection operator*.

T2. Recall problem T2 from Exercise 1.

(a) Construct the 10×10 convolution matrix A with the property that Af is the same vector than $\widetilde{p} * f$.

(b) Check that $A^T = A$.

T3. Show that $A^TA + \delta I$ is always an invertible matrix for $\delta > 0$.
Hint: note that A^TA is symmetric and study the eigenvalues of $A^TA + \delta I$.

T4. Write the following generalized Tikhonov regularized solution in stacked form:

$$\arg\min_{z \in \mathbb{R}^n} \{\|Az - m\|^2 + \delta \|Lz\|^2\}.$$

You can work on these Matlab exercises (marked with M) in the exercise session.

M1. Choose some specific vectors $\widetilde{p}$ and f for the model in Exercise T2 above. Construct the matrix A in Matlab. Compute $\widetilde{p} * f$ in two ways: by matrix multiplication and by using the Matlab routine `conv2`.

M2. Build a matrix-free version of the routine `DC5_Tikhonov_comp.m`. More precisely, solve the normal equations

$$(A^T A + \alpha I) T_\alpha(m) = A^T m$$

using GMRES. The necessary function $y \mapsto (A^T A + \alpha I)y$ can be implemented using `conv2` since $A^T = A$.

You can work on these LaTeX exercises (marked with L) in the exercise session, or you can complete them beforehand.

L1. You recently had to do the following Matlab test:

Build a for loop to the file DC6_TikhonovD_comp.m so that you plot to the same plot 100 reconstructions from data otherwise the same but with different noise every time (created by command randn without resetting the random number generator in between). Repeat the experiment with a couple of different values of regularization parameter alpha.

Now include the above three plots in your LaTeX document. More specifically, create just one Figure that contains three plots precisely on top of each other. You can use the code in the file `stackedpics.tex` provided on the course website.