Inverse problems course exam — Published March 27, 2013

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Please return your hand-written solutions to problems 1–3 and the printed solution to problem 4 in the examination session on Wednesday, April 10, 2013, at 12:15. The hall is Exactum D123.

1. Studying the course material and the literature you can find from the library, find out how the Fourier transform can be defined in the function spaces $L^p(\mathbb{R}^n)$, $p \in [1,2]$. Discuss shortly how the Fourier transform is defined in these spaces and what is the motivation of these definitions.

2. (a) Using the Fourier slice theorem, prove the formula

$$R_\omega\Big(\Delta f\Big)(s) = \frac{d^2}{ds^2}\Big(R_\omega f(s)\Big), \tag{1}$$

where $f \in C_0^\infty(\mathbb{R}^2)$ and $R_\omega f(s) = Rf(\omega, s)$, $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ is unit vector, $s \in \mathbb{R}$.

(b) Let function $u(x,t)$, $x \in \mathbb{R}^2$, $t \geq 0$ be the solution of the wave equation

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2})u(x,t) = 0, \quad x \in \mathbb{R}^2,\ t > 0,$$

$$u(x,t)|_{t=0} = \phi(x), \quad \frac{\partial}{\partial t}u(x,t)|_{t=0} = 0.$$

where $\phi \in C_0^\infty(\mathbb{R}^2)$. Using the above formula (1), consider how the Radon transform and its inverse transform can be used to write the function $u(x,t)$ in terms of the Radon transform of the function $\phi$. You can apply here the fact that the solution of the one-dimensional wave equation

$$(\partial_t^2 - \partial_s^2)w(s,t) = 0, \quad s \in \mathbb{R},\ t > 0,$$

$$w(s,t)|_{t=0} = F(s), \quad \frac{\partial}{\partial t}w(s,t)|_{t=0} = 0.$$

can be written as $w(s,t) = \frac{1}{2}(F(s-t) + F(s+t))$.

3. A problem of the form *find $f$, when $m = Af + \epsilon$ is given* is called an inverse problem if it is *not* well-posed in the sense of Hadamard.

   (a) Formulate Hadamard's definition of a well-posed problem in the context of the above inverse problem.

   (b) Write down the definition of a *regularization strategy*. Draw a picture showing the data space, the parameter space, the direct map, $m$, $f$ and the regularized inverse. Explain how the regularized solution can be understood as a noise-robust approximation of $f$.

   (c) Assume that $f \in \mathbb{R}^n$ and $\epsilon, m \in \mathbb{R}^k$ and $A \in \mathbb{R}^{k \times n}$. Prove that truncated singular value decomposition is a regularization strategy.

4. Download the file `Tikhonov_Morozov.m`. Modify the file so that it is compatible with the one-dimensional deconvolution Matlab example discussed in the course. Compute the regularized reconstruction with three clearly different noise levels. Return printed pictures and printed Matlab code in the examination session.