Finite Model Theory: Exercises (1)

Pietro Galliani

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1. Give a characterization of groups as models over a certain signature (with no function symbols) satisfying certain first-order logic formulas.

2. A group is a *periodic group* (or *torsion group*) if all of its elements have finite order (the order of an element x is the least $k \in \mathbb{N}$ such that x^k is the group's identity). Can you axiomatize periodic groups in First Order Logic?

3. Suppose that you added to First Order Logic the possibility of making *infinite disjunctions*: if Φ is a set of formulas then $\bigvee \Phi$ is another formula, and
$$\mathbf{M} \models_s \bigvee \Phi \text{ if and only if } \mathbf{M} \models_s \phi \text{ for some } \phi \in \Phi.$$
The resulting language is the *infinitary logic* $L_{\infty\omega}$.

   Can you characterize the class of all periodic groups in $L_{\infty\omega}$?

4. Suppose that $A \equiv_m A'$ (in the sense that A and A' satisfy the same formulas up to quantifier rank m) and $B \equiv_m B'$. Now let $A \uplus A'$ be the disjoint union of A and A', and let $B \uplus B'$ be the disjoint union of B and B'.

   Prove that $A \uplus A' \equiv_m B \uplus B'$.

5. Prove: For any class K of finite models over a finite signature there exists a (possibly infinite) set of first-order sentences T such that $M \models T$ if and only if $M \in K$.

6. Prove that there exists no first-order sentence ϕ, over the signature $\{P, Q\}$ (where P, Q are unary predicate symbols) such that $M \models \phi$ if and only if P^M and Q^M contain the same number of elements.