EVOLUTION AND THE THEORY OF GAMES

Model solutions 7-2-2013

4. Find all Nash equilibria (mixed and pure) of the Hawk-Dove game for $R > C$ and for $R < C$:

	H	D
H	(R-C)/2,(R-C)/2	R,0
D	0,R	R/2,R/2

$R > C$**:** (H, H) is a Nash equilibrium. There are no other pure strategy Nash equilibria.

The strategy D is strictly dominated by H. Then on the basis of exercise 7, there can not be a mixed Nash equilibrium, where D belongs to the support of the Nash. Hence there is no mixed strategy Nash equilibrium.

$R < C$**:** (D, H) and (H, D) are the only pure strategy Nash equilibria.

To find a mixed strategy Nash equilibrium, let $x = (p, 1-p)$ be the strategy of player one with p as the probability of playing H and $1-p$ as the probability of playing D. Similarly, let $y = (q, 1-q)$ be the strategy of player two.

Now

$$\pi_1(x,y) = pq\frac{R-C}{2} + p(1-q)R + (1-p)q\cdot 0 + (1-p)(1-q)\frac{R}{2}$$
$$= (1-q)\frac{R}{2} + \frac{1}{2}(R-qC)p.$$

We can see that if $R - qC > 0$, then the best response for player one is $p = 1$, which maximizes the payoff for player one. On the other hand, if $R - qC < 0$, then the best response is $p = 0$.

So the best response for player one is

$$p = \begin{cases} 1, & \text{if } q < \frac{R}{C} \\ 0, & \text{if } q > \frac{R}{C} \end{cases}$$

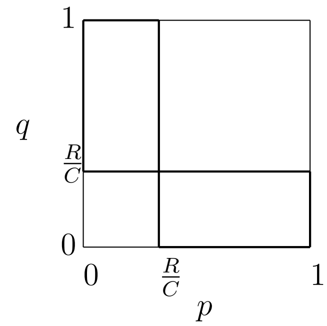
and p can be anything (between 0 and 1), if $q = \frac{R}{C}$.

Similarly we can find that for player two the best response is

$$q = \begin{cases} 1, & \text{if } p < \frac{R}{C} \\ 0, & \text{if } p > \frac{R}{C} \end{cases}$$

and q can be anything (between 0 and 1), when $p = \frac{R}{C}$.

Using the swastika-method we get the following picture:



And from the picture we see that we have a mixed Nash equilibrium $\hat{x} = \hat{y} = (\frac{R}{C}, 1 - \frac{R}{C})$.

5. Suppose that $(\hat{x}, \hat{y})$ is a Nash equilibrium. Show that $\pi_1(x, \hat{y}) = \pi_1(\hat{x}, \hat{y})$ for every pure strategy x in the support of $\hat{x}$.

Solution: Let p_i be the probability of playing pure strategy x_i ($\sum_i p_i = 1$) for the mixed strategy $\hat{x}$.

We shall prove the statement by contradiction. Assume, that the converse of the statement holds, i.e. assume that $\pi_1(x_i, \hat{y}) < \pi_1(\hat{x}, \hat{y})$ for some $x_i \in \text{supp}(\hat{x})$ (and $\pi_1(x, \hat{y}) = \pi_1(\hat{x}, \hat{y})$ for all the other $x \in \text{supp}(\hat{x})$). Since $(\hat{x}, \hat{y})$ is a Nash equilibrium, by definition it is not possible that $\pi_1(x_i, \hat{y}) > \pi_1(\hat{x}, \hat{y})$.

Now

$$\begin{aligned} \pi_1(\hat{x}, \hat{y}) &= \sum_j p_j \pi_1(x_j, \hat{y}) = p_i \pi_1(x_i, \hat{y}) + \sum_{j \neq i} p_j \pi_1(x_j, \hat{y}) \\ &< p_i \pi_1(\hat{x}, \hat{y}) + \sum_{j \neq i} p_j \pi_1(\hat{x}, \hat{y}) = \pi_1(\hat{x}, \hat{y}), \end{aligned}$$

which is a contradiction. This completes the proof.

6. Show that every dominant strategy solution is a Nash equilibrium, but that the reverse is not necessarily true.

Solution: Suppose we allow for dominant strategy solutions by the process of iterated removal of dominated strategies.
Assume that the converse holds, i.e. there is a dominated strategy solution (x_j, y_k), which is NOT a Nash equilibrium. Then there must exist a pure strategy x_i such that $\pi(x_i, y_k) > \pi(x_j, y_k)$ (or alternatively, there could be a pure strategy y_i such that $\pi(x_j, y_i) > \pi(x_j, y_k)$, but the proof is similar). But then row i can never be even weakly dominated by row j and the iterative process of eliminating dominated strategies can never end with only the strategy pair (x_j, y_k) left, which is a contradiction with (x_j, y_k) being a dominant strategy solution.

However, there can be Nash equilibria, which are not dominated strategy solutions. To see this, one can for example consider the Hawk-Dove game with $R < C$, where the strategy pairs (D, H) and (H, D) are Nash equilibria, but not dominated strategy solutions.

7. Show that if $x \in \mathbb{X}$ is a *strictly* dominated pure strategy and $(\hat{x}, \hat{y}) \in \mathbb{X} \times \mathbb{Y}$ is a Nash equilibrium, then x cannot be in the support of $\hat{x}$. Show that this conclusion need not be true if x is only *weakly* dominated. To show the latter, use the payoff matrix

	y_1	y_2	y_3
x_1	3,2	3,0	2,2
x_2	1,0	3,3	0,3
x_3	0,2	0,0	3,2

Solution: To prove the first claim by contradiction, suppose that x *is* in the support of $\hat{x}$. If x is strictly dominated by $\bar{x}$, then by the definition $\pi_1(x, y) < \pi_1(\bar{x}, y)$ for all $y \in \mathbb{Y}$, and in particular, $\pi_1(x, \hat{y}) < \pi_1(\bar{x}, \hat{y})$. But since x is in the support of $\hat{x}$, then by the Bishop-Cannings theorem $\pi_1(x, \hat{y}) = \pi_1(\hat{x}, \hat{y}) < \pi_1(\bar{x}, \hat{y})$ which contradicts the fact that $(\hat{x}, \hat{y})$ is a Nash equilibrium.

For the second part of the exercise, looking at the payoff matrix we see that (x_1, y_1), (x_2, y_2) and (x_3, y_3) are all Nash equilibria. However, the strategy y_2 (for example) is weakly dominated by y_3. Hence we have an example of a weakly dominated pure strategy, which nevertheless is in the support of a Nash equilibrium.