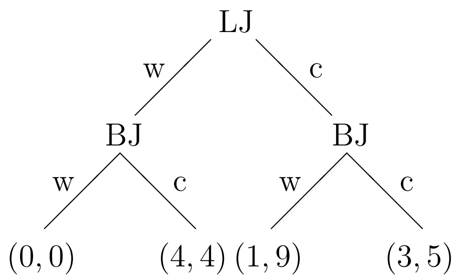
EVOLUTION AND THE THEORY OF GAMES

Model solutions 31-1-2013

1. Take the example of Big Joe and Little Joe under the banana tree (page 4 of lecture notes 01-11-2011), and solve, if possible, for dominant strategy solutions if **(a)** Little Joe makes the first move and if **(b)** both players move simultaneously. **(c)** Suppose Big Joe decides who is goint to make the first move. How would you model this situation and how would you solve it?

Solution: **(a)** In the spirit of the aforementioned example:

Game tree:



Here the payoff for LJ is listed first and the payoff for BJ second (i.e. $(1, 9)$ means a payoff of 1 for LJ and 9 for BJ).

From the tree it can be seen that, if LJ chooses to wait, then BJ is better off climbing, and if LJ chooses to climb, then BJ better wait. Assuming that BJ indeed chooses what is best for him, LJ should choose to wait in order to maximize his payoff (and then BJ will choose to climb).

The previous conclusion can be reached also in the following manner:

Strategy sets: $\mathbb{X}_{LJ} = \{w, c\}$ and $\mathbb{X}_{BJ} = \{cc, cw, wc, ww\}$. Here (for example) cw means "**c**limb if LJ waits and **w**ait if LJ climbs".

The payoff matrix:

	cc	cw	wc	ww
w	4,4	4,4	0,0	0,0
c	3,5	1,9	3,5	1,9

Here LJ is the row-player and BJ is the column-player.

From the payoff-matrix we can see that Big Joe's strategies cc, wc and ww are (at least) weakly dominated by cw. Eliminating the dominated strategies we have the reduced game:

	cw
w	4,4
c	1,9

Now Little Joe's strategy c is dominated by w and hence we end up with the dominant strategy solution $\{w, cw\}$ with a payoff $(4, 4)$.

(b) Strategy sets: $\mathbb{X}_{LJ} = \mathbb{X}_{BJ} = \{w, c\}$.

Payoff-matrix: (with LJ as the row-player and BJ the column-player)

	w	c
w	0,0	4,4
c	1,9	3,5

BJ has no strategies that are dominated since w has a higher payoff if LJ plays c (9 vs. 5) and c has a higher payoff if LJ plays w (4 vs. 0). Similarly LJ has no dominated strategies either.

There is no dominant strategy solution for this game.

(c) BJ can choose which game he wants to be played, the one where BJ chooses first or the one where LJ chooses first. We have already solved both games for a dominant strategy solution (the other one in the lecture notes and the other one on part (a) of this exercise). Therefore, in effect, BJ has to choose between either getting a payoff of 9 from the game where he chooses first or a payoff of 4 from the game where LJ chooses first. If BJ wants to maximize his payoff, he will choose to be the one who makes the first choice. Then he will choose to wait and LJ maximizes his payoff by choosing to climb.

The previous conclusion can be reached more formally.

We have the following strategy sets:

$\mathbb{X}_{BJ} = \{Bw, Bc, Lww, Lwc, Lcw, Lcc\}$, where $\{Bx\}$ means "BJ makes the first move and chooses strategy x" and $\{Lx_1x_2\}$ means "BJ lets LJ make the first move and if LJ plays w, then BJ plays x_1 or if LJ plays c, then BJ plays x_2".

$\mathbb{X}_{LJ} = \{www, wwc, wcw, wcc, cww, cwc, ccw, ccc\}$, where $\{y_1, y_2, y_3\}$ means "if BJ goes first and chooses w, then LJ chooses y_1 or if BJ goes first and chooses c, then LJ chooses y_2 or if BJ lets LJ make the first move, then LJ chooses y_3".

Now we have the following payoff matrix:

	Bw	Bc	Lww	Lwc	Lcw	Lcc
www	0,0	4,4	0,0	0,0	4,4	4,4
wwc	0,0	4,4	1,9	3,5	1,9	3,5
wcw	0,0	3,5	0,0	0,0	4,4	4,4
wcc	0,0	3,5	1,9	3,5	1,9	3,5
cww	1,9	4,4	0,0	0,0	4,4	4,4
cwc	1,9	4,4	1,9	3,5	1,9	3,5
ccw	1,9	3,5	0,0	0,0	4,4	4,4
ccc	1,9	3,5	1,9	3,5	1,9	3,5

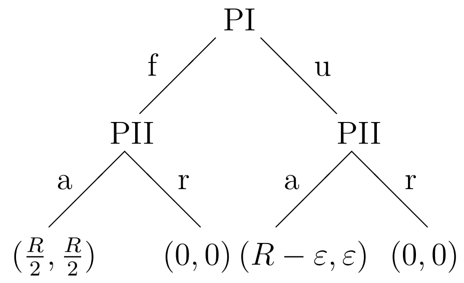
Dominant strategy solution for this payoff matrix is $\{cww, Bw\}$ with payoff $(1, 9)$. Reducing the above matrix to the dominant strategy solution is left as an exercise.

2. Suppose two players interact to decide how to divide money given to them. The first player proposes how it should be divided, and the other player either accepts the offer or rejects it. If the second player rejects, neither player gets anything.

Suppose the first player has two strategies, either to (i) propose a fifty-fifty deal (a fair deal) or (ii) an unfair deal where the second player is offered only a small positive amount (i.e. less than in the fair deal). Find a dominant strategy solution by constructing a game-tree and a payoff-matrix.

Solution: Let us denote with $R > 0$ the money to be divided and with $0 < \varepsilon < R/2$ the amount offered in the unfair deal. Player I, the one dividing the money, either offers $R/2$ if fair (strategy f), or ε if unfair (strategy u). In both cases Player II can accept (strategy a) or reject (strategy r) the offer. The strategy set for Player I is $\{f, u\}$ and for Player II $\{aa, ar, ra, rr\}$.

Game tree:



It is clear, that if Player I trusts the other player to choose a strategy that maximizes its payoff, he should choose the unfair strategy. The dominant strategy is thus $\{u, aa\}$. Player I should be unfair, and the Player II should accept every offer. The same conclusion we get by writing out the payoff matrix

	aa	ar	ra	rr
f	$\frac{R}{2}, \frac{R}{2}$	$\frac{R}{2}, \frac{R}{2}$	0,0	0,0
u	$R - \varepsilon, \varepsilon$	0,0	$R - \varepsilon, \varepsilon$	0,0

The strategies ar, ra and rr are being dominated by aa, and when eliminating these dominated strategies we have a reduced matrix where the strategy u dominates f.

3. Solve the following game, if possible, for dominant strategy solutions:

	y_1	y_2	y_3	y_4
x_1	4,5	5,3	5,6	4,4
x_2	5,3	2,1	3,5	5,2
x_3	2,6	6,3	4,2	5,5

Solution: Allowing solutions by iterated removal of dominated strategies, we proceed as follows. The strategies y_2 and y_4 are dominated by y_1. Eliminating these strategies we get the reduced game:

	y_1	y_3
x_1	4,5	5,6
x_2	5,3	3,5
x_3	2,6	4,2

Now x_3 is dominated by x_1:

	y_1	y_3
x_1	4,5	5,6
x_2	5,3	3,5

Now y_1 is dominated by y_3:

	y_3
x_1	5,6
x_2	3,5

And finally x_2 is dominated by x_1:

	y_3
x_1	5,6

And thus we have found the dominant strategy solution.