EVOLUTION AND THE THEORY OF GAMES

Model solutions 14-2-2013

8. Show that in a (symmetric) 2-by-2 payoff matrix game, the ESS conditions are equivalent to

$$\pi_1(x, x') < \pi_1(x', x')$$

or

$$\pi_1(x, x') = \pi_1(x', x') \text{ and } \pi_1(x, x) < \pi_1(x', x)$$

for every <u>*pure*</u> strategy $x \neq x'$. (Remark: x' doesn't need to be a pure strategy.)

Note that this result, if true, is quite practical. To check whether a strategy is an ESS, it is enough to test it against pure strategies!

Solution: We have to prove an "if and only if" -type of statement. The "$\Rightarrow$" direction of the proof is clear, since if the ESS-conditions hold for all strategies, they must hold also for every pure strategy.

For the "$\Leftarrow$" direction of the proof we assume that the ESS-conditions hold for every pure strategy, i.e. denoting the strategy set for both players with $\mathbb{X} = \{x_1, x_2\}$ we assume that either the first ESS condition holds

$$\pi_1(x_1, x') < \pi_1(x', x') \text{and} \tag{1}$$

$$\pi_1(x_2, x') < \pi_1(x', x') \tag{2}$$

or the second ESS condition holds

$$\pi_1(x_1, x') = \pi_1(x', x'), \tag{3}$$

$$\pi_1(x_1, x_1) < \pi_1(x', x_1) \tag{4}$$

and

$$\pi_1(x_2, x') = \pi_1(x', x') \tag{5}$$

$$\pi_1(x_2, x_2) < \pi_1(x', x_2). \tag{6}$$

Writing $x = (q, 1-q)$ and $x' = (p, 1-p)$, our task is to show that for all $x \neq x'$ $\pi_1(x, x') < \pi_1(x', x')$, or $\pi_1(x, x') = \pi_1(x', x')$ and $\pi_1(x, x) < \pi_1(x', x)$.

If the first ESS condition is satisfied for pure strategies (i.e. inequalities (1) and (2) are satisfied), then the first ESS condition is satisfied for mixed strategies x as well, because

$$\pi_1(x, x') = \sum p_i \pi_1(x_i, x') \overset{\text{ass.}}{<} \sum_i p_i \pi_1(x', x') = \pi_1(x', x'). \tag{7}$$

Now, lets check for the second ESS condition. (3) and (4) can be written as

$$\pi_1(x_1, x_1) < \pi_1(x', x_1) = p\,\pi_1(x_1, x_1) + (1-p)\,\pi_1(x_2, x_1)$$
$$\Longleftrightarrow$$
$$\pi_1(x_1, x_1) < \pi_(x_2, x_1).$$

Similarly from (5) and (6) we get that $\pi_1(x_2, x_2) < \pi_1(x_1, x_2)$.

From equalities (3) and (5) we can solve p

$$\pi_1(x_1, x) = p\,\pi_1(x_1, x_1) + (1-p)\,\pi_1(x_1, x_2) = p\,\pi_1(x_2, x_1) + (1-p)\,\pi_1(x_2, x_2)$$
$$= \pi_1(x_2, x)$$

which gives:

$$p = \frac{\pi_1(x_1, x_2) - \pi(x_2, x_2)}{\pi_1(x_1, x_2) - \pi_1(x_2, x_2) + \pi_1(x_2, x_1) - \pi_1(x_1, x_1)}. \tag{8}$$

Clearly $p \in (0, 1)$, i.e. p is a probability since $\pi_1(x_1, x_2) > \pi(x_2, x_2)$ and $\pi_1(x_2, x_1) > \pi_1(x_1, x_1)$.

From here on we shall denote:

$$\begin{aligned} a &= \pi_1(x_1, x_1), \\ b &= \pi_1(x_1, x_2), \\ c &= \pi_1(x_2, x_1) \text{ and} \\ d &= \pi_1(x_2, x_2). \end{aligned}$$

Now

$$\begin{aligned} \pi_1(x, x) - \pi_1(x', x) = {} & q^2 a + q(1-q)b + (1-q)qc + (1-q)^2 d - pqa \\ & - p(1-q)b - (1-p)qc - (1-p)(1-q)d. \end{aligned} \tag{9}$$

One can now open up the expressions, see that some terms cancel, regroup the terms and use the fact that $(b-d) = p(b-d+c-a)$ (this we get from equation (8)). After this fairly straightforward, but laborious manipulation of equation (9) we finally arrive at

$$\pi_1(x, x) - \pi_1(x', x) = -(q-p)^2(b-d+c-a) < 0,$$

which concludes the proof.

9. Calculate all evolutionarily stable strategies (pure and mixed) for the Hawk-Dove game

	H	D
H	$\frac{1}{2}R - \frac{1}{2}C,\ \frac{1}{2}R - \frac{1}{2}C$	$R, 0$
D	$0, R$	$\frac{1}{2}R, \frac{1}{2}R$

for **(a)** $R > C$ **(b)** $R = C$ and **(c)** $R < C$.

Solution: **(a)** We have

$$\pi(H,H) = \frac{R-C}{2} > 0 = \pi(D,H)$$

and

$$\pi(D,D) = \frac{V}{2} < V = \pi(H,D),$$

which shows that H is an ESS if we consider pure strategies only. What about mixed strategies? Consider mixed strategy $x = (p, 1-p)$, where p is the probability of playing H. Now

$$\pi(x,H) = p\frac{R-C}{2} + (1-p)\cdot 0 = p\frac{V-C}{2} < \pi(H,H)$$

and therefore H is still an ESS also if mixed strategies are considered. Since the support of one ESS may not be a subset of another, there can be no mixed ESS. The only ESS is H.
Note that we could have alternatively used the result from exercise 8, i.e. it would be enough to show that H is an ESS if tested against pure strategies only; the mixed strategy condition follows automatically for 2x2 matrices. That is, if $\pi(H,H) > \pi(D,H)$ then $\pi(H,H) > \pi(x,H)$ for all mixed $x \neq (1,0)$.

(b) With only pure strategies we have

$$\pi(H,H) = 0 = \pi(D,H)$$

and

$$\pi(D,D) = \frac{R}{2} < V = \pi(H,D),$$

so the first ESS-condition fails, but the second is valid and therefore H is an ESS. D is not an ESS just as before.

We can use again the result from exercise 8 to conclude that H is an ESS also if mixed strategies are concerned. Because the support of one ESS may not be a subset of another, there can be no mixed ESS.

(c) Now

$$\pi(H,H) = \frac{1}{2}(R-C) < 0 = \pi(D,H)$$

and

$$\pi(D,D) = \frac{R}{2} < V = \pi(H,D),$$

so there is no pure strategy ESS.

Let $x = (p, 1-p)$. Now

$$\pi(H,x) = \frac{R-C}{2}p + R(1-p)$$

and

$$\pi(D,x) = 0 \cdot p + \frac{R}{2}(1-p).$$

The Bishop-Cannings theorem tells us that if x is an ESS, the following must hold:

$$\pi(H,x) = \pi(D,x) = \pi(x,x).$$

From this we get

$$\frac{R-C}{2}p + R(1-p) = \frac{R}{2}(1-p).$$

Solving for p we find that

$$p = \frac{R}{C} \in (0,1).$$

Again, since the Hawk-Dove game is a symmetric 2-by-2 matrix game we can use exercise 8 to conclude that x is an ESS if $\pi(x,H) > \pi(H,H)$ and $\pi(x,D) > \pi(D,D)$. Indeed, these conditions hold:

$$\pi(x,H) = \frac{1}{2}(R-C)\frac{R}{C} > \frac{1}{2}(R-C) = \pi(H,H)$$

(remember that $R - C < 0$) and

$$\pi(x,D) = \frac{1}{2}R(\frac{R}{C} + 1) > \frac{1}{2}R = \pi(D,D).$$

10. Extend the Hawk-Dove game with a third strategy called "Bully" (B) who plays Hawk against Dove but Dove against Hawk, and also Dove against itself. Give the payoff matrix of the Hawk-Dove-Bully game and calculate all ESSs for **(a)** $R > C$ and **(b)** $R < C$.

Solution: The payoff matrix:

	H	D	B
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	$R, 0$	$R, 0$
D	$0, R$	$\frac{1}{2}R, \frac{1}{2}R$	$0, R$
B	$0, R$	$R, 0$	$\frac{1}{2}R, \frac{1}{2}R$

(a) We first look at only pure strategies. Since $\pi(D, H) < \pi(H, H)$ and $\pi(B, H) < \pi(H, H)$, then H is an ESS. On the other hand, $\pi(D, D) < \pi(H, D)$ and $\pi(B, B) < \pi(H, B)$ and therefore D and B are not ESSs.

Is H an ESS if we take mixed strategies into account as well? Let $x = (p, q, 1 - p - q)$. Then

$$\pi(x, H) = p\frac{R-C}{2} < \frac{R-C}{2} = \pi(H, H)$$

and therefore H is an ESS.

Since H is an ESS, there can be no mixed strategy ESS, which has a positive probability of playing H (the support of one ESS may not be a subset of another). However, there could still be a mixed ESS of the form $y = (0, q, 1 - q)$. But here the Bishop-Cannings condition $\pi(D, y) = \pi(B, y)$ yields a contradiction $\frac{R}{2} = 0$ (we assume $R \neq 0$).

The only ESS is H.

(c) None of the pure strategies is a best reply to itself, so there is no pure strategy ESS. Let's consider a mixed strategy with full support. We denote this strategy as $x = (p, q, 1 - p - q)$. The Bishop-Cannings condition $\pi(D, x) = \pi(B, x)$ yields a contradiction $p = 1$.

Let's consider the mixed strategy $y = (p, 0, 1-p)$. The Bishop-Cannings condition $\pi(H, x) = \pi(B, x)$ yields $p = \frac{R}{C} \in (0, 1)$. Is $y = (\frac{R}{C}, 0, 1 - \frac{R}{C})$ an ESS? Let $z = (a, b, 1-a-b)$. Since $\pi(z, y) - \pi(y, y) = -\frac{R}{2}[b(1 - \frac{R}{C})]$, the first ESS condition is satisfied for all $b \neq 0$, but for $b = 0$ we need to check the second ESS condition, i.e. whether $\pi(z, z) - \pi(y, z) < 0$. We get $\pi(z, z) - \pi(y, z) = \frac{(R-Ca)^2}{2}) < 0$ for all $a \neq R/C$, i.e. for all $z \neq y$ and hence the second ESS condition is satisfied. We have that $y = (\frac{R}{C}, 0, 1 - \frac{R}{C})$ is an ESS. From the Hawk-Dove game we also know that a mixed strategy $(\frac{R}{C}, 1 - \frac{R}{C}, 0)$ is an ESS. Note that here we have two mixed ESS where the supports overlap. This is no contradiction since they are not subsets of each other!