EVOLUTION AND THE THEORY OF GAMES

Model solutions 28-2-2013

11. Generalize the Bishop-Cannings theorem: Let $X \subset R$ be an interval, and let $F : X \to R$ be a distribution function representing a mixed strategy over X with some support. Show that if F is an ESS, then $\pi_1(x, F) = \pi_1(F, F)$ for all x in the support of F, where π is continuous with respect to x.

Solution: If F is an ESS, then $\pi_1(x, F) \leq \pi_1(F, F)$, and so if the claim is not true then there must exist x_0 in the support of F such that $\pi_1(x_0, F) < \pi_1(F, F)$. By continuity, there exist $a, b \in X$ such that

$$\pi_1(x_0, F) < \pi_1(F, F) \quad \text{for all } x \in [a, b] \tag{1}$$

and

$$\int_a^b f(x)dx > 0, \tag{2}$$

where f is the density function of F. We get

$$\begin{aligned}
\pi_1(F, F) &= \int_0^\infty \pi_1(x, F)f(x)dx \\
&= \int_0^a \pi_1(x, F)f(x)dx + \int_a^b \pi_1(x, F)f(x)dx + \int_b^\infty \pi_1(x, F)f(x)dx \\
&< \int_0^a \pi_1(F, F)f(x)dx + \int_a^b \pi_1(F, F)f(x)dx + \int_b^\infty \pi_1(F, F)f(x)dx \\
&= \pi_1(F, F)
\end{aligned}$$

which is a contradiction and proves the claim.

12. Let F and G be two different distribution function over some interval $x \subset R$. Show that if F and G are both ESSs, then the support of one cannot be a subset of the other (and *vice versa*).

Solution: Let us reach a contradiction by assuming that there exists some other ESSs G such that its support is a subset of the support of F. Then by the generalized Bishop-Cannings Theorem we get

$$\pi_1(x, F) = \pi_1(F, F) \tag{3}$$

for all x in the support of G and so $\pi_1(G,F) = \pi_1(F,F)$. But since G is an ESS as well, and the 1st ESS condition is apparently not satisfied, then the 2nd ESS condition $\pi_1(G,F) > \pi_1(F,F)$ must be. But this contradicts F to be an ESS, which proves the claim.

13. The 'Size Game' is related to the War of Attrition, but in contrast to the latter, the cost of displaying is paid before the game starts and is not refunded. Does the Size Game have **(a)** (2 points) a pure ESS or **(b)** (4 points) a mixed ESS with full support?

Solution: **(a)** For every pure strategy c_1 there exist another pure strategy $c_2 = c_1 + \varepsilon$, where $0 < \varepsilon < R/2$, for which $\pi_1(c_2, c_1) > \pi_1(c_1, c_1)$. Hence there are no pure ESSs.

(b) Let us see if we can find a mixed strategy ESS with full support. Let F be the cumulative distribution function of a mixed strategy, and suppose that F is everywhere continuously differentiable. Let $f = F'$ be the probability density function of F. Then the payoff for a pure strategy c against F is

$$\pi_1(c,F) = (R-c)\int_0^c f(\gamma)d\gamma - c\int_c^\infty f(\gamma)d\gamma \tag{4}$$

$$= R\int_0^c f(\gamma)d\gamma - c\int_0^\infty f(\gamma)d\gamma \tag{5}$$

$$= R\int_0^c f(\gamma)d\gamma - c \tag{6}$$

If F is an ESS, then by the Bishop-Cannings theorem necessarily $\pi_1(c,F) = \pi_1(F,F)$. Differentiation with respect to c yields

$$Rf(c) - 1 = 0 \rightarrow f(c) = \frac{1}{R} \tag{7}$$

But now f is not a probability distribution since integrating it over $[0,\infty]$ does not equal one. Therefore there is no mixed ESS with full support.

Note that this does not guarantee that there could not exist ESSs with partial support.

14. Who takes care of the kids? Suppose a male has two possible strategies: he can be faithful and help the female with taking care of the offspring, or he can philander and abandon the female right after mating. A female also has two possible strategies: she can be coy by demanding a long courtship period before mating or she can be fast by skipping the courtship. Philandering males and coy females dont get along and do not mate, but other combinations of males and females do mate. The reward of producing offspring is $R/2$ per player. The cost of rearing the offspring is C, which is either shared by the parents if the male is faithful, or which is borne totally by the female otherwise. The cost of courtship is d to both the male and the female. Give the payoff matrix and analyze the game as an asymmetric game.

Solution: The payoff matrix:

	coy	*fast*
faithful	$\frac{1}{2}(R-C)-d,\ \frac{1}{2}(R-C)-d$	$\frac{1}{2}(R-C),\ \frac{1}{2}(R-C)$
philander	$0, 0$	$\frac{1}{2}R, \frac{1}{2}R-C$

Here the male player has the row strategies and female column strategies.
(faithful,coy) is an ESS if and only if

$$\left\{ \begin{array}{l} \frac{1}{2}(R-C)-d>0 \\ \frac{1}{2}(R-C)-d>\frac{1}{2}(R-C) \end{array} \right. \tag{8}$$

The second condition reduces to $d<0$, which is not a reasonable value for d. We conclude that (faithful,coy) is never an ESS.

(philander,coy) is an ESS if and only if

$$\left\{ \begin{array}{l} \frac{1}{2}(R-C)-d<0 \\ \frac{1}{2}(R-C)<0 \end{array} \right. \tag{9}$$

that is,

$$\left\{ \begin{array}{l} d>\frac{1}{2}(R-C) \\ C>\frac{1}{2}R \end{array} \right. \tag{10}$$

(faithful,fast) is an ESS if and only if

$$\left\{ \begin{array}{l} \frac{1}{2}(R-C)>\frac{1}{2}R \\ \frac{1}{2}(R-C)>\frac{1}{2}(R-C)-d \end{array} \right. \tag{11}$$

The first condition reduces to $C<0$, which is not a reasonable value for C. We conclude that (faithful,fast) is never an ESS.

(philander,fast) is an ESS if and only if

$$\left\{ \begin{array}{l} \frac{1}{2}R>\frac{1}{2}(R-C) \\ \frac{1}{2}R-C>0 \end{array} \right. \tag{12}$$

that is,

$$\begin{cases} C > 0 \\ \frac{1}{2}R > C \end{cases} \tag{13}$$

In summary:
If $R > 2C$, then (philander,fast) is the only ESS.
If $R < 2C$ and $d > \frac{1}{2}(R - C)$, then (philander,coy) is the only ESS.
If $R < 2C$ and $d < \frac{1}{2}(R_C)$ then there is no ESS.

Note that in the first ESS case the female has a kid and raises it by herself, and in the second ESS case they don't have offspring. There is no ESS where the offspring would be raised by both parents.

15. Solve the Hawk-Dove-Assessor game when the Assessor knows its own rank regarding, say, strength within the population as a whole and assuming that the stronger player always wins in a H×H-contest. How does the ESS depend on the cost of assessment and on the Assessor's own rank?

Solution: : Let $k \in [0, 1]$ be the normalized ranking of an individual, i.e. k is the probability to be stronger than the opponent, i.e. of an Assessor (denoted by A) playing Hawk. Let $x = (p, 1 - p)$ be a mixed strategy for choosing Hawk or Dove. Then

$$\pi_1(x, A) = k\pi(x, H) + (1 - k)\pi(x, D) = p(\frac{1}{2}R - \frac{1}{2}Ck) + \frac{1}{2}R(1 - k). \tag{14}$$

As it is a linear equation, it obtains a maximum at $p = 0$ when $k > \frac{R}{C}$ and at $p = 1$ when $k < \frac{R}{C}$. Then, for $k > \frac{R}{C}$ an A is an ESS if $\pi_1(A, A) > \pi_1(D, H)$, i.e. when

$$k > \frac{2(R + \gamma)}{3R + C} \tag{15}$$

and for $k < \frac{R}{C}$ an A is an ESS if $\pi_1(A, A) > \pi_1(D, H)$, i.e. when

$$k > \frac{1}{3} + \frac{2\gamma}{3R}. \tag{16}$$