EVOLUTION AND THE THEORY OF GAMES

Model solutions 21-3-2013

18. Consider the Hawk-Dove game with payoff matrix

	H	D
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	$R, 0$
D	$0, R$	$\frac{1}{2}R, \frac{1}{2}R$

(a) (3 points) Analyze the *iterated* Hawk-Dove game for the strategies 'always dove' (allD) and 'always hawk' (allH) , i.e. calculate the payoff matrix of the iterated game and determine whether and when any of the two strategies is an ESS.

(b) (3 points) Idem (i.e. do as above: calculate the payoff matrix and give ESS conditions) for allH, allD and Bully (B), which starts with H in the first round, but then does the opposite of what its opponent did in the previous round. (Note that the payoff matrix is a 3×3-matrix)

(c) (3 points) Idem for allH, allD and Retaliator (R), which starts with D in the first round, but then does the same as what its opponent did in the previous round. (as above, the payoff matrix is a 3×3-matrix)

(d) (3 points) Idem for the strategies Bully (B) and Retaliator (R).

Solution: As usual, we assume $R > 0$ and $C > 0$.

(a) The payoff for an allH-player against an allH-player is:

$$E_{HH} = \frac{1}{2}(R-C) + \delta E_{HH}$$

Solving this for E_{HH} gives:

$$E_{HH} = \frac{\frac{1}{2}(R-C)}{1-\delta}.$$

The other payoffs are calculated similarly to get the following payoff matrix:

	allH	allD
allH	$\frac{\frac{1}{2}(R-C)}{1-\delta}$, $\frac{\frac{1}{2}(R-C)}{1-\delta}$	$\frac{R}{1-\delta}$, 0
allD	0 , $\frac{R}{1-\delta}$	$\frac{\frac{1}{2}R}{1-\delta}$, $\frac{\frac{1}{2}R}{1-\delta}$

From the payoff matrix we see that:
allH is an ESS

$$\Leftrightarrow \frac{\frac{1}{2}(R-C)}{1-\delta} > 0$$
$$\Leftrightarrow R > C$$

allD is an ESS

$$\Leftrightarrow \frac{\frac{1}{2}R}{1-\delta} > \frac{R}{1-\delta}$$
$$\Leftrightarrow \text{never (we're assuming } R > 0)$$

Above we show that allH satisfies the first ESS-condition if $R > C$. What if $R = C$? Then π(allD,allH) $= 0 =$ π(allH,allH) and the first ESS-condition fails. Then we should look at the second ESS-condition, which gives π(allD,allD) $<$ π(allH,allD) and therefore allH is an ESS also if $R = C$. If $R < C$ both ESS-conditions fail.

From now on we shall ignore the second ESS condition: this is for simplicity, but it can also be argued that the only loss is one line in the parameter space $R > 0, C > 0$. That is, we calculated which strategy is an ESS (if any) for all values of R, C except on the line $R = C$. Unless the model has some restrictions, the probability that the value of the resource is exactly the value of the cost is 0.

(b) As an example, we calculate the payoff for a Bully playing against another Bully. The players go into a $(H \times H)$ - $(D \times D)$ -cycle. Then we have

$$\begin{cases} E_{HH} = \frac{1}{2}(R-C) + \delta E_{DD} \\ E_{DD} = \frac{1}{2}R + \delta E_{HH} \end{cases}$$
$$\Rightarrow E_{HH} = \frac{1}{2}(R-C) + \delta(\frac{1}{2}R + \delta E_{HH})$$
$$\Rightarrow E_{HH} = \frac{\frac{1}{2}(R-C) + \frac{\delta}{2}R}{1-\delta^2}$$

After calculating all the other payoffs similarly we get the following payoff matrix:

	allH	allD	B
allH	$\frac{\frac{1}{2}(R-C)}{1-\delta}$, $\frac{\frac{1}{2}(R-C)}{1-\delta}$	$\frac{R}{1-\delta}$, 0	$\frac{1}{2}(R-C)+\frac{\delta R}{1-\delta}$, $\frac{1}{2}(R-C)$
allD	0 , $\frac{R}{1-\delta}$	$\frac{\frac{1}{2}R}{1-\delta}$, $\frac{\frac{1}{2}R}{1-\delta}$	0 , $\frac{R}{1-\delta}$
B	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)+\frac{\delta R}{1-\delta}$	$\frac{R}{1-\delta}$, 0	$\frac{\frac{1}{2}(R-C)+\frac{\delta}{2}R}{1-\delta^2}$, $\frac{\frac{1}{2}(R-C)+\frac{\delta}{2}R}{1-\delta^2}$

The strategy allH is an ESS if and only if

$$\begin{cases} \dfrac{\frac{1}{2}(R-C))}{1-\delta} > 0 \\ \dfrac{\frac{1}{2}(R-C))}{1-\delta} > \frac{1}{2}(R-C), \end{cases}$$

which holds if and only if $R > C$.

The strategy allD is never an ESS (as in part (a)).

The strategy B is an ESS if and only if

$$\begin{cases} \dfrac{\frac{1}{2}(R-C)+\frac{\delta}{2}R}{1-\delta^2} > 0 \\ \dfrac{\frac{1}{2}(R-C)+\frac{\delta}{2}R}{1-\delta^2} > \frac{1}{2}(R-C)+\dfrac{\delta R}{1-\delta}, \end{cases}$$

but given the assumptions $R > 0$, $C > 0$ and $0 < \delta < 1$ the latter inequality does not hold (this may be a tedious task to do by hand, but can be easily verified using computer software such as Mathematica or Maple) and therefore B is never an ESS.

(c) The payoffs are calculated similarly as before (and in the lecture notes). The payoff matrix is:

	allH	allD	R
allH	$\frac{\frac{1}{2}(R-C)}{1-\delta}$, $\frac{\frac{1}{2}(R-C)}{1-\delta}$	$\frac{R}{1-\delta}$, 0	$R+\frac{\frac{\delta}{2}(R-C)}{1-\delta}$, $\frac{\frac{\delta}{2}(R-C)}{1-\delta}$
allD	0 , $\frac{R}{1-\delta}$	$\frac{\frac{1}{2}R}{1-\delta}$, $\frac{\frac{1}{2}R}{1-\delta}$	$\frac{\frac{1}{2}R}{1-\delta}$, $\frac{\frac{1}{2}R}{1-\delta}$
R	$\frac{\frac{\delta}{2}(R-C)}{1-\delta}$, $R+\frac{\frac{\delta}{2}(R-C)}{1-\delta}$	$\frac{\frac{1}{2}R}{1-\delta}$, $\frac{\frac{1}{2}R}{1-\delta}$	$\frac{\frac{1}{2}R}{1-\delta}$, $\frac{\frac{1}{2}R}{1-\delta}$

The strategy allH is an ESS if and only if

$$\begin{cases} \dfrac{\frac{1}{2}(R-C)}{1-\delta} > 0 \\ \dfrac{\frac{1}{2}(R-C)}{1-\delta} > \dfrac{\frac{\delta}{2}(R-C)}{1-\delta}, \end{cases}$$

which holds if and only if $R > C$.

The strategy allD is never an ESS just as before.

The strategy R is never an ESS. From the payoff matrix we see that π(allD,R) $= \frac{\frac{1}{2}R}{1-\delta} = \pi$(R,R), so the first ESS-conditions fails. We also see that π(R,allD) $= \frac{\frac{1}{2}R}{1-\delta} = \pi$(allD,allD), so the second ESS-condition fails as well.

(d) *Bully vs. Retaliator:* Here the contest becomes an $(H \times D)$ - $(H \times H)$ - $(D \times H)$ - $(D \times D)$ -cycle. We have the following set of equations (for the Bully):

$$\begin{aligned} E_{HD} &= R + \delta E_{HH} \\ E_{HH} &= \frac{1}{2}(R-C) + \delta E_{DH} \\ E_{DH} &= 0 + \delta E_{DD} \\ E_{DD} &= \frac{1}{2}R + \delta E_{HD}. \end{aligned}$$

Solving for E_{HD} gives:

$$E_{HD} = \frac{R + \frac{\delta}{2}(R-C) + \delta^3 \frac{R}{2}}{1-\delta^4}.$$

The payoff for Retaliator against a Bully is calculated similarly. The $B \times B$ and $R \times R$ -contest we have already calculated in the previous parts of the exercise. The payoff matrix thus is:

	B	R
B	$\frac{\frac{1}{2}(R-C)+\frac{\delta}{2}R}{1-\delta^2}, \frac{\frac{1}{2}(R-C)+\frac{\delta}{2}R}{1-\delta^2}$	$\frac{R+\frac{\delta}{2}(R-C)+\delta^3\frac{R}{2}}{1-\delta^4}, \frac{\frac{\delta}{2}(R-C)+\delta^2 R+\frac{\delta^3}{2}R}{1-\delta^4}$
R	$\frac{\frac{\delta}{2}(R-C)+\delta^2 R+\frac{\delta^3}{2}R}{1-\delta^4}, \frac{R+\frac{\delta}{2}(R-C)+\delta^3\frac{R}{2}}{1-\delta^4}$	$\frac{\frac{1}{2}R}{1-\delta}, \frac{\frac{1}{2}R}{1-\delta}$

The strategy B is an ESS if and only if

$$\frac{\frac{1}{2}(R-C) + \frac{\delta}{2}R}{1-\delta^2} > \frac{\frac{\delta}{2}(R-C) + \delta^2 R + \frac{\delta^3}{2}R}{1-\delta^4}.$$

Again this looks rather ugly, but simplifying this inequality with Mathematica, it reduces to:

$$R > \frac{C - \delta C + \delta^2 C}{1-\delta^2}.$$

The strategy R is an ESS if and only if

$$\frac{\frac{1}{2}R}{1-\delta} > \frac{R + \frac{\delta}{2}(R-C) + \delta^3\frac{R}{2}}{1-\delta^4}.$$

Using Mathematica, this condition simplifies to:

$$R < \frac{\delta C}{1-\delta^2}.$$

19. Consider two bulls fighting over the right to mate with the females of the herd. The game goes like this: First there is a lot of display of power and intention. Next, the two males run to one another on a collision course. Each can either swerve (S) or not swerve (NS). If neither swerves, both males end up with a terrible headache (or worse), and neither one wins. If both swerve, then there is no winner either, but at least no-one got hurt, and the game may be repeated again some other time. If only one of the males swerves, the other is the winner and gains the right to mate with all the females.

The prize of winning is R; the cost of display and charging is D; the cost of the headache (or worse) is C. the payoff matrix of the game thus is:

	S	NS
S	$-D$, $-D$	$-D$, $R-D$
NS	$R-D$, $-D$	$-C-D$, $-C-D$

(a) (3 points) Analyze this game as a single shot (i.e. not repeated) game. Calculate the mixed ESS? What is the role of the "participation fee" D?

(b) (3 points) Analyze the game as an iterated game in which the game is repeated with a probability $\delta \in (0,1)$ and *only if both males swerved.* Consider only the pure strategies swerve (S) (or 'allS'), not swerve (NS) (or 'allNS') and the mixed strategy (M) found in (a). What is the role of the "participation fee" D now?

(c) (3 points) Do you think that there are better mixed strategies than M? Analyze a game similar to a game in (b), but now only with the pure strategies swerve (S) and not swerve (NS) (and as above, game is repeated only if both males swerved). Find a mixed ESS (G) and compare it to (M).

Solution: Again, we have $R > 0$ and $C > 0$.

(a) S is an ESS if and only if $-D > R - D$, which only holds if $R < 0$. Therefore S is never an ESS.

NS is an ESS if and only if $-C - D > -D$, which only holds if $C < 0$. Therefore NS is never an ESS.

Let us look for a mixed ESS. let $x = (p, 1-p)$ be a mixed strategy. Is x is an ESS, then by the Bishop-Cannings theorem we have $\pi(S,x) = \pi(NS,x)$, which gives:

$$-pD - (1-p)D = p(R-D) - (1-p)(C+D)$$
$$\Leftrightarrow \; p = \frac{C}{R+C}$$

To show that $x = (\frac{C}{R+C}, 1 - \frac{C}{R+C})$ is an ESS for such a game as this, it is sufficient to show that $\pi(x,S) > \pi(S,S)$ and $\pi(x,NS) > \pi(NS,NS)$ (see proposition in the Lecture notes). Indeed, these conditions do hold (as can be easily verified) and therefore x is an ESS. Since it has full support, it is also the only ESS.

The participation fee D is paid by all the contestants and therefore it does not play any role in the game other than to make the whole game less profitable. In fact, this game can be analyzed easier if we add D to all the payoffs. This will cancel all the D's and the payoff matrix will be just

	S	NS
S	0, 0	0, R
NS	R, 0	$-C$, $-C$

(b) The payoff matrix:

	allS	allNS	M
allS	$-\frac{D}{1-\delta}, -\frac{D}{1-\delta}$	$-D,\ R-D$	$\pi(S,M),\ \pi(M,S)$
allNS	$R-D,\ -D$	$-C-D,\ -C-D$	$-D,\ -(1-p)C-D$
M	$\pi(M,S),\ \pi(S,M)$	$-(1-p)C-D,\ -D$	$\pi(M,M),\ \pi(M,M)$

where

$$\pi(S,M) = -\frac{pD}{1-\delta} - (1-p)D$$
$$\pi(M,S) = -\frac{pD}{1-\delta} + (1-p)(R-D)$$

and

$$\pi(M,M) = -\frac{D[C^2 + 2CR(1-\delta) + R^2(1-\delta)]}{(C+R)^2(1-\delta)}.$$

allS and allNS are not ESSs (which can be verified as in part (a)). M is an ESS if and only if $\pi(M,M) > \pi(S,M)$ and $\pi(M,M) > \pi(NS,M)$. Again the equations for these are somewhat ugly, but simplifying them with Mathematica gives that the conditions hold if

$$\delta < \frac{R^2}{CD + R^2}.$$

Now the participation fee does play a role as the allS-players have to pay it repeatedly. Consequently, the mixed ESS (M) found in part (a) is not an ESS anymore for this game with the following payoff matrix:

	allS	allNS
allS	$-\frac{D}{1-\delta}, -\frac{D}{1-\delta}$	$-D, R-D$
allNS	$R-D, -D$	$-C-D, -C-D$

This can be seen by verifying that the Bishop-Cannings condition does not hold for M.

(c) The question here is whether for a game similar to one in (b) there exist a better mixed strategy than the mixed strategy M found in (a). In (b) we found that M is not an ESS in that game, so if there exist a mixed ESS it will do better than M (by definition).

Let us look for the mixed ESS $G = (p, 1-p)$, where with probability p strategy S is played.

The Bishop-Cannings condition gives us a candidate-ESS:

$$p = \frac{C}{R + C + D(\frac{1}{1-\delta} - 1)}.$$

It is straightforward to verify that $\pi(G, allS) > \pi(allS, allS)$ and $\pi(G, allNS) > \pi(allNS, allNS)$. Therefore, G indeed is an ESS. Since it has full support, it is also the only ESS.

Comparing strategies M and G we see that $p_M > p_G$, i.e. the probability to play S is higher for M. We can conclude that in the game where S has to pay the participation fee D for every round, it is better to play S more infrequently (as compared to the game where D is paid only once). This is not surprising as the punishment for playing S is greater for iterated games, then the mixed strategy ESS in an iterated game should have a smaller probability to play S.