EVOLUTION AND THE THEORY OF GAMES

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Contents

Lecture 16-1-2013

1. Introduction

The course "Evolution and the theory of games" is part of the biomathematics curriculum at the Department of Mathematics and Statistics of the University of Helsinki. The focus is on animal behavior: can we understand animal behavior from the theory of games?

Parts of this course are taken from the books *"Game theory evolving: a problem-centered introduction to modeling strategic interaction'* by Herbert Gintis (2009, 2nd edition, Princeton University Press) and *"Evolution and the theory of games'* by John Maynard Smith (1982, Cambridge University Press).

A game is a model of a situation of conflicts of interests where the pay-off to one player depends not only on his own strategy but also on the strategies of the other players. The phrase "*but also on the strategies of the other players*" makes that a game is not a mere optimization problem.

For example, the Traveling Salesman Problem (of finding the shortest route through a finite number of towns) is an optimization problem. However, if there are several salesmen and a bonus is given for arriving in a town first (presumably because who comes first strikes the best deal), then the problem is no longer a problem of mere optimization because the best route is not only short but also takes into account the choices of the others.

What games do animals play? What counts as a *solution* of a game, and how do we find it? These are questions we address first by means of examples.

1.1. **Prisoner's dilemma.**

The *prisoner's dilemma* (PD): Two men have been sentenced to serve two years in prison for the possession of illegal recreational substances. It is suspected, however, that they are not just users but also dealers. If proven true, this will give them two extra years in prison. Each prisoner is offered the following deal: if one gives testimony against the other, he gets one year reduction (for helping the police) and the other gets two extra years (because now there is proof that he is a dealer). The prisoner's are not allowed to talk to one another and so have to make their choices independently.

What would the prisoners chose: testify against the other or not? The payoffs depend on their combined choices as given in the following matrix:

	not testify	testify
not testify	-2, -2	-4, -1
testify	-1, -4	-3, -3

The payoffs are given as negative the number of years to serve in prison. The first of each pair is the payoff to the row-player (on the left) and the second of each pair is the payoff to the column-player (at the top).

It can be seen that if the column-player chooses not to testify, the row-player better does testify as this gives him only one year in prison rather than the two years he serves if he does not testify. If instead the column-player choses to testify, then the row-player also better testifies, because in that way he serves only three years instead of the four he serves if he does not testify.

So, whatever the choice of the column-player, the row-player better testifies. By symmetry the same holds for the column player, and both players end up testifying against the other with the result that each serves three years in prison. The strategy 'testify' is said to dominate the strategy 'not testify', and the outcome (both players testify) is called a dominant strategy solution. And this is the paradox: by attempting to minimize the time in prison, both players end up with a sub-optimal outcome: if both refuse to testify against the other they get only two years in prison.

The prisoner's dilemma is used as a basic model for the evolution of cooperation among humans or animals. Note that the prisoner's dilemma does not explain cooperation: it predicts non-cooperation (i.e., between the players, not with the police), which raises the question why there exists cooperation at all. Later we shall see that if the prisoner's dilemma is played repeatedly between the same players, under some circumstances cooperation will be the outcome.

The canonical form of the prisoner's dilemma is given by the payoff matrix

	C	D
C	R, R	S, T
D	T, S	P, P

where C stands for 'cooperation' (with the other player) and D for 'defection' (i.e., non-cooperation). The payoffs are denoted by T (*temptation*), R (*reward*), P (*punishment*) and S (*sucker's payoff*) and are ordered as $S < P < R < T$. One readily checks that D is a dominant strategy, i.e. gives a higher payoff than the alternative irrespective of the choice of the opponent.

Do animals play the prisoner's dilemma? Animal grooming (i.e., cleaning the fur or skin of one individual by removing parasites and dirt by another individual) may be an example of the PD in nature. Let c denote the cost of grooming someone else, and let b denote the benefit of being groomed yourself, and assume that the benefit exceeds the cost, i.e., $b > c > 0$. The payoff matrix then is

	C	D
C	$b-c,\ b-c$	$-c,\ b$
D	$b,\ -c$	$0,\ 0$

where C now stands for *grooming* and D for *not grooming*. It can be seen immediately that the payoff matrix has the ordering of a prisoner's dilemma game. Other examples have been given in the lecture.

1.2. **Hawk-Dove game.** The *hawk-dove game* (HD) describes a contest between two individuals for a resource like a food item or a territory and is used as a basic model for the evolution of aggression. There are two strategies: *hawk* (H) and *dove* (D). Someone playing H will fight for the resource until someone gives up or gets hurt. Someone playing D will not fight and either give up or share the resource evenly if the other player also has D. Here is the payoff matrix:

	H	D
H	$\frac{1}{2}R - \frac{1}{2}C,\ \frac{1}{2}R - \frac{1}{2}C$	$R,\ 0$
D	$0,\ R$	$\frac{1}{2}R,\ \frac{1}{2}R$

where R is the value of the resource and C is the cost of injury. The factor $\frac{1}{2}$ appears because each player in a H$\times$H-contest wins or looses with a probability one-halve, and likewise, in a D$\times$D-contest, each player gets the resource with a probability one-halve.

If $R > C$ (i.e., the value of the resource outweighs the potential cost of injury), then H is a dominant strategy which is preferred above D no matter how the opponent choses to play. The outcome of the game therefore is that both players chose H. This is the dominant strategy solution of the game. And here we see the same paradox as with the prisoner's dilemma: if both players chose H, their payoff is less than if both chose D.

If $R < C$, there is no dominant strategy. To find a 'solution' of the game (in some sense or another) we have to develop another solution concept than the dominant strategy solution. We do that later; first we develop a more formal language to describe games in general.

Remarks. Another version of a hawk-dove game is called the 'chicken game', where two drivers drive towards each other on a collision course. If neither of them swerves and moves away, they crash. However, the one who swerves first will be called chicken, and receive contempt from the viewers. This version of the game is also in the scenes of the movies 'Footloose' and 'Rebel without a cause'. The game of chicken has been famously compared to a nuclear brinkmanship by the philosopher Bertrand Russell.

1.3. **Characterization of games.** A *two-person game* is fully characterized by the following three things:

(1) The strategy sets X and Y, one for each player but not necessarily the same.
(2) The payoff function $\pi : X \times Y \to \mathbb{R}^2$ where $\pi(x, y)$ is the pair of payoffs to the first and second players if they choose the strategies $x \in X$ and $y \in Y$.
(3) The solution concept (like the 'dominant strategy solution' as in the previous examples, but there are other solution concepts too).

(An N-person game has N players with strategy sets $X_1, \ldots, X_N$ and payoff function $\pi : X_1 \times \cdots \times X_N \to \mathbb{R}^N$.)

In the example of the prisoner's dilemma we have $X = Y = \{\mathrm{C}, \mathrm{D}\}$. The payoff function is given by the entries of the payoff matrix, and so

$$\begin{cases} \pi(\mathrm{C},\mathrm{C}) & = & (R, R) \\ \pi(\mathrm{C},\mathrm{D}) & = & (S, T) \\ \pi(\mathrm{D},\mathrm{C}) & = & (T, S) \\ \pi(\mathrm{D},\mathrm{D}) & = & (P, P) \end{cases}$$

The solution concept we used was the 'dominant strategy solution', which gives us the strategy pair (D,D).

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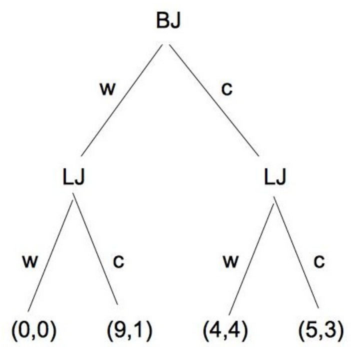
Example (Big Joe & Little Joe) Before we go into alternative solution concepts, have a look at the following example taken from *"Game theory evolving"* by Herbert Gintis (but with bananas instead of coconuts):

Big Joe and Little Joe are two monkeys sitting under a banana tree. The idea is that at least one of them has to climbs up and knocks the bananas down so that they can be eaten. The dilemma is this: if BJ climbs up while LJ waits below, then LJ can already start eating before BJ is back on the ground, and *vice versa*.

The costs and benefits are the following: The total energetic value of the bananas is 10 kcal. The cost of climbing is 2 kcal for BJ and 0 kcal for LJ, who is the better climber. If both climb the tree, then after they're back on the ground, BJ gets a share worth of 7 kcal while LJ gets only 3 kcal. If BJ climbs the tree while LJ waits down below, this will be 6 kcal for BJ versus 4 kcal for LJ. If BJ waits and LJ climbs, this becomes 9 kcal versus 1 kcal. If both wait, they don't get any bananas, and so both get 0 kcal.

There are various possible scenarios, depending on which player makes the first move, leading to different games with different strategy sets and different payoff functions. Here we consider the case where BJ moves first; the other possibilities are left as an exercise.

We may represent the game as a so-called game tree (see figure): at the top is the 'root' indicated by BJ, who decides either to wait (w) or to climb (c) as shown by the first two branches. Once BJ has made his move, LJ will have to make his move as shown by the second-level branches. At the bottom of the figure are the payoffs to BJ (first of each pair) and to LJ (second of each pair) depending on their combined moves (and taken into account that BJ is a lousy climber).



From the tree it can be seen that if BJ chooses to wait, then LJ better climbs, and if BJ chooses to climb then LJ better waits. Assuming that LJ indeed chooses what is best for him, BJ should choose to wait in order to maximize his payoff.

How to model this game in the way presented in the previous section? If BJ moves first, his strategy set is simply $X_{\mathrm{BJ}} = \{\mathrm{w,c}\}$: wait or climb. However, LJ can use the information about BJ's first move, i.e., he can use conditional strategies: 'if BJ waits I do this and if BJ climbs I do that'. His strategy set therefore is $X_{\mathrm{LJ}} = \{\mathrm{ww,wc,cw,cc}\}$, where the first letter of each pair tells what LJ does if BJ waits while the second letter tells what LJ does if BJ climbs. For example, 'cw' means 'if BJ waits, I climb, and if BJ climbs, I wait'. The payoff function is given by the payoff matrix

	cc	cw	wc	ww
w	9,1	9,1	0,0	0,0
c	5,3	4,4	5,3	4,4

where BJ is the row-player and LJ the column player. It can be seen that the payoff to strategy cw is always better (or at least not worse) than the other strategies of LJ irrespective BJ's: in other words cw is a dominant strategy. Eliminating the other strategies of LJ, we obtain a reduced game with the payoff matrix

	cw
w	9,1
c	4,4

In this reduced game, the strategy w is a dominant strategy for BJ, and so we end up with the dominant strategy solution (w,cw), which is also what we found by inspection of the game tree.

Remarks: (1) The payoffs for (w,cw) are the same as the payoffs for (w,cc), but cw and cc are essentially different strategies, which becomes immediately clear if you compare the payoffs for (c,cw) and (c,cc). (2) The strategy cw is strongly dominating wc but only weakly dominating cc and ww, because some of the payoffs are the same. The solution (w,cw) is therefore also called a weakly dominant strategy solution. (3) Note that w is not a dominant strategy in the first payoff matrix, i.e., before we reduced the size of the game by eliminating the (weakly or strongly) dominated strategies for LJ.

2. Solution concepts

2.1. **Dominated strategy.** In the previous lecture we introduced the notion of a 'game' as a model of a situation of conflict where the payoff to one player depends not only on his own strategy but on the strategies of the other players as well. In particular, a 2-person game was defined by the following three things:

(1) Two strategy sets X and Y.

(2) A payoff function $\pi : X \times Y \to \mathbb{R}^2$.

(3) A solution concept.

Modeling a situation of conflict of interests thus amounts to choosing (1) the strategy sets, (2) a payoff function and (3) a solution concept in a way that best suits the situation. It should be emphasized that these are modeling choices; they do not follow in any way from the mathematics.

So far we have seen one solution concept: the dominant strategy solution. The dominant strategy is defined as follows

Definition Consider two strategy sets X and Y for players I and II, respectively. A strategy $x \in X$ is dominated by strategy $x' \in X$ if

$$\pi(x, y) \leq \pi(x', y) \quad \forall y \in Y \tag{1}$$

and if

$$\exists y' \in Y \quad \text{s.t.} \quad \pi(x, y') < \pi(x', y'). \tag{2}$$

In words, a strategy $x \in X$ is said to be dominated by $x' \in X$ if $\pi_1(x, y) \leq \pi_1(x', y)$ for all $y \in Y$ with a strict inequality for at least one value of y. If the inequality is strict for all y we speak of 'strong' domination, otherwise of 'weak' domination. We have seen how removal of dominated strategies can lead to a single strategy pair, which we then call a dominated strategy solution.

The example of Big Joe and Little Joe under the banana tree showed that once we have eliminated all dominated strategies for each player, it may turn out that a strategy that was not dominated at the onset now is dominated among the strategies that remain. And so we may be able to remove dominated strategies in several rounds. This is called the method of iterated removal of dominated strategies.

It should be kept in mind, however, that the game is *not played* in rounds; the elimination of dominated strategies in one or more sequential steps happens only in the mind of the players before they choose their strategy and therefore before the game is played. The dominant strategy solution thus presumes a rather sophisticated level of rationality for all players. The following game requires the players to think four rounds ahead:

	y_1	y_2	y_3	y_4
x_1	4,5	5,3	5,6	4,4
x_2	5,3	2,1	3,5	5,2
x_3	2,6	6,3	4,2	5,5

The actual solving of this game is left as an exercise.

Sometimes there is *no* dominant strategy solution. Take the following example:

	y_1	y_2	y_3
x_1	1,4	2,1	$4,1\frac{1}{2}$
x_2	2,1	4,4	3,2

There is no dominated strategy, and so the elimination method stalls already in the first round.

2.2. **Mixed strategies.** In the above example, we could enforce a solution by either changing the solution concept, or extending the strategy space. Let's do the latter by adding the mixed strategy $y^* = \frac{1}{2}y_1 + \frac{1}{2}y_2$ (i.e., play y_1 or y_1, each with a probability of one-half) to the strategy set of the column-player:

	y_1	y_2	y_3	y^*
x_1	1,4	2,1	$4,1\frac{1}{2}$	$1\frac{1}{2}, 2\frac{1}{2}$
x_2	2,1	4,4	3,2	$3, 2\frac{1}{2}$

Now y_3 is dominated by y^* and therefore can be eliminated, which gives

	y_1	y_2	y^*
x_1	1,4	2,1	$1\frac{1}{2}, 2\frac{1}{2}$
x_2	2,1	4,4	$3, 2\frac{1}{2}$

Now x_1 is dominated by x_2 and can be removed:

	y_1	y_2	y^*
x_2	2,1	4,4	$3, 2\frac{1}{2}$

Now both y_1 and y^* are dominated by y_2 and can be eliminated, and so we end up with the dominant strategy solution (x_2,y_2).

Note that y^* does not occur in the solution and yet was needed to complete the elimination procedure. Actually, this is quite absurd, and maybe we should have included mixed strategies from the beginning. Taking mixed strategies aboard, however, presumes something about the rationality of the players, namely, that they are actually able to generate random numbers, an assumption that may or may not be warranted depending on the context.

Definition A *mixed strategy* is a probability distribution over a given set of pure strategies. Given the pure strategies $x_1, \ldots, x_k$, we can represent a mixed strategy by a vector of probabilities $p_1, \ldots, p_k$ where $p_i \geq 0$ $\forall i$ and $\sum p_i = 1$ and where p_i is the probability of playing strategy x_i.

Definition The set $\{x_i : \ p_i > 0, \ i = 1, \ldots, k\}$ is called the *support* of the mixed strategy.

Remarks: (1) The notion of mixed strategy as a probability distribution is readily generalized to countably many pure strategies or even a continuum of pure strategies. (2) Mixed strategy can be represented as a linear combination of pure strategies $\sum_i p_i x_i$ (3) The (expected) payoff for players $i = 1, 2$ playing mixed strategies x and y, resp., is given by $\pi_i(x, y) = \sum_j \sum_k p_j q_k \pi_i(x_j, y_k)$, where pure strategies x_j and y_k are being played by players $i = 1, 2$ with probabilities p_j and q_k, resp..

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If we allow for mixed strategies, the existence of a dominant strategy solution is still not guaranteed. Take the Hawk-Dove game:

	H	D
H	$\frac{1}{2}R - \frac{1}{2}C$, $\frac{1}{2}R - \frac{1}{2}C$	R, 0
D	0, R	$\frac{1}{2}R$, $\frac{1}{2}R$

If $R > C$, then (H,H) is the dominant strategy solution, but with $R < C$ there is no dominant strategy solution, neither pure (which is obvious) nor mixed. To see that there is no mixed solution, let $x = (p, 1-p)$ and $y = (q, 1-q)$ be mixed strategies where p and q are the probabilities of playing Hawk. Then

$$\begin{aligned}\pi_1(x, y) &= \tfrac{1}{2}(R-C)pq + Rp(1-q) + 0(1-p)q + \tfrac{1}{2}R(1-p)(1-q) \\ &= \tfrac{1}{2}R(1-q) + \tfrac{1}{2}(R-qC)p\end{aligned}$$

So, π_1 is an increasing function of p if $q < R/C$ but a decreasing function of p if $q > R/C$. Hence, there do not exist any two values p and p' such that one gives a higher payoff *for all* q, because the ordering of the respective payoffs is reversed if we change from $q < R/C$ to $q > R/C$.

The dominant strategy solution is a very credible solution concept, but it need not exist. Moreover, the presumed level of rationality of the players may limit its usefulness in the context of animal behavior. There exist many other solution, but each has its pros and contras. Some of these alternatives we have a look at now.

2.3. **More solution concepts.**

Definition The *Hicks optimum* maximizes total payoff as a function of the strategies of all players.

For example, the Prisoner's Dilemma with $T > R > P > S$ and $2R > T + S$ has a unique Hicks optimum, namely (C,C).

In the context of the evolution of animal behavior, accepting the Hicks optimum would amount to accepting solutions *for the good of the species* or, on a smaller scale, for the good of the population. Such a solution, however, is not stable against cheaters: in the Prisoner's Dilemma, for example, a defector would enjoy a higher payoff than the cooperators, and so the *temptation* for a player to defect destroys the coherence of the group and makes the Hicks optimum (C,C) an unlikely outcome.

Definition The *Pareto optimum* (or Pareto efficient) is defined such that every possible change of strategies (for one or more players) is disadvantageous for at least one of the players.

For example, the Prisoner's Dilemma with $T > R > P > S$ has three Pareto optima: (C,C), (D,C) and (C,D), but again *none of these is free of the temptation* for one or both players to change their strategy from cooperation to defection. With the Pareto optimum the emphasis shifts to pre-game negotiation and contract, which in the context of animal behavior may not be very realistic.

Remarks: (1) A change of strategies which doesn't make anyone worse off and makes at least one player better off is called *Pareto improvement.* Pareto optimum is achieved when no improvements are possible (2) Pareto optimum is said to be *socially efficient.*

Definition The *minimax solution* is such that each player chooses a strategy that minimizes his maximum loss, which is equivalent to maximizing one's minimum payoff.

In the Hawk-Dove game, the minimum payoff to Hawk is $\frac{1}{2}(R-C)$, and the minimum payoff to Dove is 0:

	H	D	$\min\{\pi_1\}$
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	R, 0	$\frac{1}{2}(R-C)$
D	0, R	$\frac{1}{2}R$, $\frac{1}{2}R$	0
$\min\{\pi_2\}$	$\frac{1}{2}(R-C)$	0	

If $R > C$, then for each player the minimum payoff is maximized by choosing Hawk, and so the minimax solution is (H,H). If $R < C$, then for each player the minimum payoff is maximized by choosing Dove, and so the minimax solution is (D,D). In the latter case, however, both player are *tempted* to change unilaterally from D to H, because Hawk against Dove gives a higher payoff.

2.4. **Nash equilibrium.**

Definition A *Nash equilibrium* in a two-person game with strategy sets X and Y is a strategy point $(\hat{x}, \hat{y}) \in X \times Y$ such that

$$\begin{cases} \pi_1(x,\hat{y}) \le \pi_1(\hat{x},\hat{y}) & \forall x \in X \\ \pi_2(\hat{x},y) \le \pi_2(\hat{x},\hat{y}) & \forall y \in Y \end{cases}$$

Remarks: (1) The definition of a Nash equilibrium readily generalizes to an N-person game (see below) (2) A strategy $\hat{x} \in X$ is said to be a 'best response' to a strategy $y \in Y$ if it maximizes the players expected payoff, i.e. $\pi_1(x,y) \le \pi_1(\hat{x},y) \quad \forall x \in X$ (3) Nash equilibrium in other words: $\hat{x}$ and $\hat{y}$ are best responses to one another, and hence neither player cannot increase his/hers payoff by changing strategy. (4) The *absence of temptation* to change one's strategy is build into the definition of the Nash equilibrium.

For example, take the Hawk-Dove game: if $R > C$, then (H,H) is a Nash equilibrium, and if $R < C$, then (D,C) and (C,D) are Nash equilibria. As a next example, consider

	y_1	y_2
x_1	0,2	2,1
x_2	3,1	1,3

which does not have a Nash equilibrium of pure strategies. The question is whether there is an equilibrium with mixed strategies.

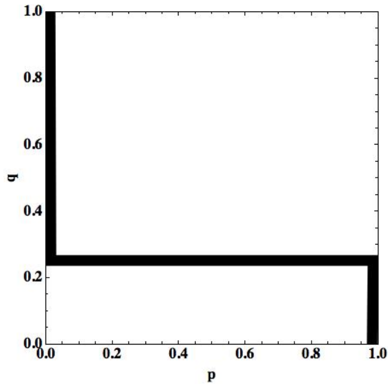
Theorem. Every game with finitely many pure strategies for each of the N players has at least one Nash equilibrium if mixed strategies are allowed.

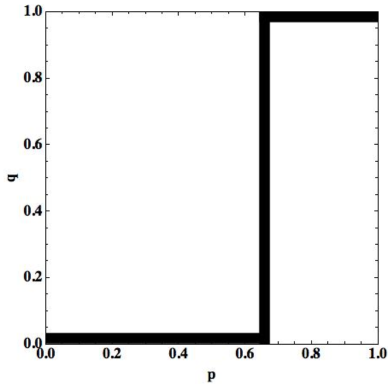
The proof is given in the next lecture.

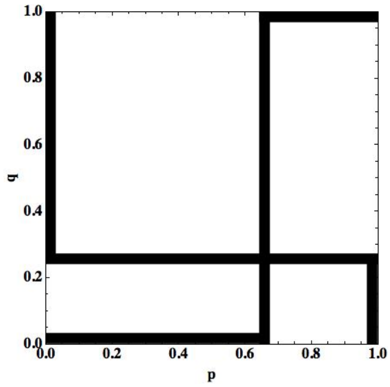
The theorem implies that the last example must have a Nash equilibrium which necessarily involves mixed strategies. To find this solution, let $x = (p, 1-p)$ and $y = (q, 1-q)$ be mixed strategies. The payoff to the row-player is $\pi_1(x,y) = 1 + 2q + (1-4q)p$, and so the best response to $q < \frac{1}{4}$ is $p = 1$, and the best response to $q > \frac{1}{4}$ is $p = 0$, while for $q = \frac{1}{4}$ any $p \in [0,1]$ is a best response, as indicated by the thick line in the next figure:

The payoff to the column-player is $\pi_1(x,y) = 3 - 2p + (3p-2)q$, and so the best response to $p < \frac{2}{3}$ is $q = 0$, and the best response to $p > \frac{2}{3}$ is $q = 1$, and for $p = \frac{2}{3}$ any $q \in [0,1]$ is a best response:

Since the first figure gives the set of best responses to q and the second figure the best responses to p, their superposition gives at the intersection of the thick lines the pair of strategies $(\hat{p},\hat{q}) = (\frac{2}{3},\frac{1}{4})$ that are best responses to one another and hence is a Nash equilibrium:







For obvious reasons this graphical method is called the *Swastika method*. Unfortunately, this method is only practical for two-person games with two pure strategies for each player. Next lecture we introduce a more general method.

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Definition A Nash equilibrium in an N-person game with strategy sets $\Sigma_1, \ldots, \Sigma_N$ is a strategy combination $(\hat{\sigma}_1, \ldots, \hat{\sigma}_N) \in \Sigma_1 \times \cdots \times \Sigma_N$ such that

$$\pi_i(\hat{\sigma}_1, \ldots, \sigma_i, \ldots, \hat{\sigma}_N) \leq \pi_i(\hat{\sigma}_1, \ldots, \hat{\sigma}_N) \quad \forall \sigma_i \in \Sigma_i$$

for all $i \in \{1, \ldots, N\}$.

Remark: Note that σ_j's are mixed strategies.

Definition A game is said to have a *finite strategic form* when each player has a finite number of (pure) strategies and the number of players is finite.

If in addition the game cannot go on indefinitely the game is said to be *finite* (Note that some authors, e.g. Nash, define finite games as we defined finite strategic games).

Theorem (*Nash existence theorem*). Every game that has a finite strategic form has at least one Nash equilibrium in mixed strategies $\Box$

For the proof we shall need the following result:

Theorem (*Brouwer's fixed point theorem*). Let S be a subset of $\mathbb{R}^n$ that is closed, bounded and convex, and let f be a continuous function. Then f has at least one fixed point, that is, a point s in S so that $f(s) = s$. $\Box$

We don't prove Bouwer's fixed point theorem, but make sure you understand what it says: the meaning of a set being closed and bounded is clear. A set $C \in \mathbb{R}^n$ is convex if for every $c_1, c_2 \in C$ and $0 \leq p \leq 1$ the weighed average $pc_1 + (1-p)c_2 \in C$. Hence, the set of all mixed strategies over a finite set of pure strategies is convex.

Proof (*Nash existence theorem*). The proof is for the case $N = 2$ but is readily generalized to any number of players. First some notation and definitions:

Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ be pure strategies for, respectively, the row- and column-player, and let $x = (p_1, \ldots, p_m) \in X$ and $y = (q_1, \ldots, q_n) \in Y$ be mixed strategies. Define the functions

$$\begin{aligned} c_i(x,y) &= \max\{0, \pi_1(x_i, y) - \pi_1(x,y)\} \\ d_i(x,y) &= \max\{0, \pi_2(x, y_i) - \pi_2(x,y)\} \end{aligned}$$

and

$$f(x,y) = \begin{pmatrix} f_X(x,y) \\ \\ f_Y(x,y) \end{pmatrix} = \begin{pmatrix} \frac{p_1+c_1(x,y)\ ,\ \dots\ ,\ p_m+c_m(x,y)}{1+\sum_i c_i(x,y)} \\ \\ \frac{q_1+d_1(x,y)\ ,\ \dots\ ,\ q_m+d_n(x,y)}{1+\sum_i d_i(x,y)} \end{pmatrix}$$

Thus, map f shifts each probability p_i, q_j by c_i, d_j and then normalizes it so that the mapped points become probabilities as well.

We claim that f has a fixed point $(\hat{x}, \hat{y}) \in X \times Y$ and this fixed point is a Nash equilibrium.

To see that f has a fixed point, note that X and Y are convex, and so is $X \times Y$. Moreover, $f_X(x,y)$ and $f_Y(x,y)$ are probability distributions over $x_1, \dots, x_m$ and $y_1, \dots, y_n$ (check this!), and so f takes values in $X \times Y$. Finally, f is continuous, and so we can apply Brouwer's fixed point theorem.

To see that a fixed point of f is a Nash equilibrium, we prove that $\pi_1(x, \hat{y}) \leq \pi_1(\hat{x}, \hat{y})\ \forall x \in X$. (The proof of $\pi_2(\hat{x}, y) \leq \pi_2(\hat{x}, \hat{y})\ \forall y \in Y$ is similar and therefore will not be given.)

Write $\hat{x} = (\hat{p}_1, \dots, \hat{p}_m)$, and take $i_0 \in \{1, \dots, m\}$ such that $c_{i_0}(\hat{x}, \hat{y}) = 0$. Note that such i_0 exists, because if it did not, then $c_i(\hat{x}, \hat{y}) > 0\ \forall i$, i.e., $\pi_1(x_i, \hat{y}) > \pi_1(\hat{x}, \hat{y})\ \forall i$, and so $\pi_1(\hat{x}, \hat{y}) = \sum_i \hat{p}_i \pi_1(x_i, \hat{y}) > \sum_i \hat{p}_i \pi_1(\hat{x}, \hat{y}) = \pi_1(\hat{x}, \hat{y})$, which is a contradiction.

Since $(\hat{x}, \hat{y}) = f(\hat{x}, \hat{y})$, we have in particular

$$\hat{p}_{i_0} = \frac{\hat{p}_{i_0} + c_{i_0}(\hat{x}, \hat{y})}{1 + \sum_i c_i(\hat{x}, \hat{y})} = \frac{\hat{p}_{i_0}}{1 + \sum_i c_i(\hat{x}, \hat{y})}$$

and so $\sum_i c_i(\hat{x}, \hat{y}) = 0$. Since by definition the c_i are non-negative, it follows that $c_i(\hat{x}, \hat{y}) = 0$, i.e., $\pi_1(x_i, \hat{y}) \leq \pi_1(\hat{x}, \hat{y})\ \forall i$. Hence, for arbitrary $x = (p_1, \dots, p_m) \in X$ we have

$$\pi_1(x, \hat{y}) = \sum_i p_i \pi_1(x_i, \hat{y}) \leq \sum_i p_i \pi_1(\hat{x}, \hat{y}) = \pi_1(\hat{x}, \hat{y})$$

which is what we set out to prove. □

Remarks: Above is the proof from *Nash, J. 1951: Non-cooperative games. Annals of Mathematics 54, pp. 286-295.* Nash published a proof year before (in *Nash, J. 1950: Equilibrium points in N-person games. Proc. Nat. Acad. Sci. USA 36. pp. 48-49*) using Kakutanis fixed point theorem. However, quoting Nash in his 1951 paper, the proof given here is a considerable improvement over that earlier version.

In the previous lecture we have seen that the game with payoff matrix

	y_1	y_2
x_1	0,2	2,1
x_2	3,1	1,3

does not have a Nash equilibrium of pure strategies. However, the Nash existence theorem tells us that there is a Nash equilibrium if we allow for mixed strategies. We used the Swastika method to find out what that equilibrium is. The Swastika method, however, is practical for 2×2-payoff matrices only. Here is a result that can be used more generally:

Theorem (*Bishop-Cannigns theorem*). If $(\hat{x}, \hat{y})$ is a Nash equilibrium, then $\pi_1(x, \hat{y}) = \pi_1(\hat{x}, \hat{y})$ for every pure strategy x in the support of $\hat{x}$, and $\pi_2(\hat{x}, y) = \pi_2(\hat{x}, \hat{y})$ for every pure strategy y in the support of $\hat{y}$. □

The proof is given as an exercise. To see how the Bishop-Cannings theorem is used, consider the popular Rock-Paper-Scissors game with payoff matrix:

	R	P	S
R	0 , 0	-1 , 1	1 , -1
P	1 , -1	0 , 0	-1 , 1
S	-1 , 1	1 , -1	0 , 0

Obviously there is no *pure strategy* Nash equilibrium: the best reply to Rock is Paper, and the best reply to Paper is Scissors, and the best reply to Scissors is Rock, and so there is no pair of strategies that are best replies to each other. Still, from the Nash existence theorem we know that there does exist at least one *mixed strategy* Nash equilibrium.

We first look for a solution $(\hat{x}, \hat{y})$ with full support for both strategies. From the Bishop-Cannings theorem we have

$$\pi_1(R, \hat{y}) = \pi_1(P, \hat{y}) = \pi_1(S, \hat{y}) = \pi_1(\hat{x}, \hat{y})$$

Written out for $\hat{y} = (q_1, q_2, q_3)$ and the given payoff matrix this becomes

$$-q_2 + q_3 = +q_1 - q_3 = -q_1 + q_2 = \pi_1(\hat{x}, \hat{y})$$

which, together with $q_1 + q_2 + q_3 = 1$, form a system of four independent linear equations in q_1, q_2, q_3 and $\pi_1(\hat{x}, \hat{y})$. Solving these equations gives

$$q_1 = q_2 = q_3 = \frac{1}{3} \quad \& \quad \pi_1(\hat{x}, \hat{y}) = 0$$

Similarly, for $\hat{x} = (p_1, p_2, p_3)$ we find

$$p_1 = p_2 = p_3 = \frac{1}{3} \quad \& \quad \pi_2(\hat{x}, \hat{y}) = 0$$

The conditions in the definition of the Nash equilibrium are readily verified to show that $(\hat{x}, \hat{y})$ is a Nash equilibrium, indeed.

In fact, the above equilibrium is also the only equilibrium of the game. The proof of this is left as an exercise.

Lecture 4-2-2013

There are two known examples of the Rock-Paper-Scissors game in nature. The first is found in a species of Californian lizards (*Uta Stansburiana*). The males come in three types: (1) the ultra-dominant polygynous orange-throated males,(2) the mate-guarding monogamous blue-throated males, and (3) the female-mimicking yellow-throated males (these males look similar to sexually mature females, and also, their declining to mate -behavior mimicks the female behavior). Orange wins against Blue but looses against Yellow, because the latter looks like a female and can sneak into the harem of Orange to mate with the females. Yellow, however, looses against the Blue, because the latter guards a single female against any intruder. So, the best reply to Orange is Yellow, and the best reply to Yellow is Blue, and the best reply to Blue is Orange. (*For further details see the paper by Sinervo and Livley in* Nature*(1996) 340: 240-243.*)

The second example is found in *E. coli*, the common gut bacteria. There are three types: (1) type C produces a toxin called colicin but is itself not affected, (2) type R is resistant to colicin but cannot make it, and (3) type S is susceptible and gets killed when exposed to colicin. Type R grows faster than type C and outcompetes it in an initially mixed culture, because it does not have the cost of producing the toxin. Type S grows even faster than R, because it does not have the cost of developing and maintaining a detoxification mechanism. Type C, however, invades and outcompetes a S, because the latter has no defense against the toxin. So R beats C, C beats S, and S beats R.

In spite of its charms, the Nash equilibrium also has its problems: e.g., take the Hawk-Dove game

	H	D
H	$\frac{1}{2}R - \frac{1}{2}C$, $\frac{1}{2}R - \frac{1}{2}C$	R, 0
D	0, R	$\frac{1}{2}R$, $\frac{1}{2}R$

with $R < C$ so that (H,D) and (D,H) are Nash equilibria. But how is such an outcome realized? The Hawk gets the resource and the Dove gets nothing. Note that the game is played only ones, so why would one player voluntarily take the role of Dove? Without some form of coordination there is a high chance that the players end up with (H,H) or (D,D). In other words, accepting the Nash equilibria (H,D) and (D,H) as credible solutions of the game shifts the emphasis from the game to pre-game negotiations and contract. In the context of animal behavior, such negotiations may or may not be realistic, but this is the reason why we introduce one more (and last) solution concept.

2.5. **Evolutionarily stable strategy.**

Definition An *evolutionarily stable strategy* (ESS) is a strategy such that, if adopted by a sufficiently large fraction of the population, then no other strategy can invade, i.e., increase in frequency (*Maynard Smith 1982*).

Note that "invasion of one strategy in a population of an other strategy" is changing population densities over time and hence about population dynamics. So, it should not come as a surprise that the *exact mathematical conditions* that make a strategy an ESS depend on the population dynamical embedding of the game. There are various ways how to do this, but the following is by far the most common:

Let $x, x' \in X$ be two strategies that occur in the population with relative frequencies ε and $1-\varepsilon$, and assume that opponents are assigned randomly so that the probability of being paired with an x-player is ε, and the probability of being paired with a x'-player is $1-\varepsilon$. Then the expected payoff to an x-player is

$$\varepsilon\,\pi_1(x,x) + (1-\varepsilon)\pi_1(x,x')$$

and to an x'-player is

$$\varepsilon\,\pi_1(x',x) + (1-\varepsilon)\pi_1(x',x')$$

(Note that the constancy of being paired with a player of a given type implies an infinitely large population or "sampling with replacement".)

Suppose further that x cannot increase in relative frequency if and only if x-players are "doing worse" than x'-players in terms of their expected payoff, i.e., if

$$\varepsilon\,\pi_1(x,x) + (1-\varepsilon)\pi_1(x,x') < \varepsilon\,\pi_1(x',x) + (1-\varepsilon)\pi_1(x',x')$$

Collecting terms with the same order of ε, this can be rewritten as

$$\big[\pi_1(x,x') - \pi_1(x',x')\big] + \varepsilon\big[\pi_1(x,x) - \pi_1(x',x) - \pi_1(x,x') + \pi_1(x',x')\big] < 0$$

For sufficiently small ε the sign of the left hand side is dominated by the sign of $\pi_1(x,x') - \pi_1(x',x')$. (i.e. for all $x, x' \in X$ for which $\pi_1(x,x') = \pi_1(x',x')$ there exists an ε such that $|\varepsilon\big[\pi_1(x,x) - \pi_1(x',x) - \pi_1(x,x') + \pi_1(x',x')\big]| < |[\pi_1(x,x') - \pi_1(x',x')|$). Only when this term is zero, the sign of the remaining order-ε term $\pi_1(x,x) - \pi_1(x',x)$ matters.

Hence, x' is evolutionarily stable (in the context of random pairing with replacement) if and only if for every $x \neq x'$ we have either

$$\pi_1(x,x') < \pi_1(x',x') \quad (\text{"1}^{\text{st}}\text{ ESS condition"})$$

or

$$\pi_1(x,x') = \pi_1(x',x') \;\&\; \pi_1(x,x) < \pi_1(x',x) \quad (\text{"2}^{\text{nd}}\text{ ESS condition"})$$

Remarks:

(1) Often the above conditions are given as the definition of an ESS, but in fact this is suitable only in the case of random pairing of opponents with replacement.

(2) If x' is an ESS, then $\pi_1(x, x') \leq \pi_1(x', x')$ $\forall x$, and so the strategy pair (x', x') is a (symmetric) Nash equilibrium.

(3) Note every symmetric Nash equilibrium corresponds to an ESS, because the ESS conditions are more restrictive.

(4) Like the two strategies of a Nash equilibrium (x, y) are best responses to one another, so is an ESS a best response to itself.

(5) Since an ESS corresponds to a Nash equilibrium, the Bishop-Cannings theorem applies: if x' is an ESS, then $\pi_1(x, x') = \pi_1(x', x')$ for every pure strategy x in the support of x'.

Let us look at some examples.

Consider the *Prisoner's Dilemma* with payoff matrix

	C	D
C	R, R	S, T
D	T, S	P, P

with $T > R > P > S$. Then

$$\pi_1(\mathrm{C}, \mathrm{D}) = S < P = \pi_1(\mathrm{D}, \mathrm{D})$$

which satisfies the first ESS condition, and so D is an ESS. However,

$$\pi_1(\mathrm{D}, \mathrm{C}) = T > R = \pi_1(\mathrm{C}, \mathrm{C})$$

which violates both ESS conditions, and so C is not an ESS.

We use the Bishop-Cannings theorem to find possible mixed ESSs. For example, consider the question *Who takes care of the kids?* There are two strategies: stay (S) and run (R). A parent who stays contributes to the cost of protecting the brood and feeding the hatchlings. If both parents stay, the costs are equally divided. However, a parent who runs away has none of these costs, while the one who stays pays all. If both parents run away, the offspring does not survive.

Let C be the total cost of raising the offspring, and let V be the value of the offspring (in terms of 'fitness') per breeding pair. The payoff matrix then is

	S	R
S	$\frac{1}{2}V - \frac{1}{2}C$, $\frac{1}{2}V - \frac{1}{2}C$	$\frac{1}{2}V - C$, $\frac{1}{2}V$
R	$\frac{1}{2}V$, $\frac{1}{2}V - C$	0 , 0

We further assume that $V > 2C$ so that there is no pure ESS. To see whether there is a mixed ESS, write $x = (p, 1-p)$ where $p \in (0,1)$ is the probability of staying. Applying the Bishops-Cannings theorem we get

$$\begin{cases} \pi_1(\mathrm{S}, x) &= p\big(\frac{1}{2}V - \frac{1}{2}C\big) + (1-p)\big(\frac{1}{2}V - C\big) &= \pi_1(x,x) \\ \pi_1(\mathrm{R}, x) &= p\,\frac{1}{2}V + (1-p)\,0 &= \pi_1(x,x) \end{cases}$$

and so

$$p\big(\frac{1}{2}V - \frac{1}{2}C\big) + (1-p)\big(\frac{1}{2}V - C\big) = p\,\frac{1}{2}V$$

from which we find

$$p = \frac{V-2C}{V-C} \in (0,1)$$

But is $x = (p, 1-p)$ an ESS? By construction (i.e., how we calculated p) the first ESS condition does not hold, and so we have to check the second condition. To this end the following proposition is quite useful:

Proposition. For the second ESS condition to hold it is necessary and sufficient that $\pi_1(\mathrm{S},\mathrm{S}) < \pi_1(x,\mathrm{S})$ and $\pi_1(\mathrm{R},\mathrm{R}) < \pi_1(x,\mathrm{R})$. In other words, we only have to check that the second ESS condition holds for *pure strategies*. □

The proof is left as an exercise. Applying the proposition to $x = (p, 1-p)$ with $p = (V-2C)/(V-C)$ gives

$$\pi_1(\mathrm{S},\mathrm{S}) - \pi_1(x,\mathrm{S}) = -\frac{C^2}{2(V-C)} < 0$$

and

$$\pi_1(\mathrm{R},\mathrm{R}) - \pi_1(x,\mathrm{R}) = -p\left(\frac{1}{2}V - C\right) < 0$$

and so x is an ESS indeed.

Lecture 6-2-2013

Every two-person game with finitely many strategies has a Nash equilibrium if mixed strategies are allowed. Does a similar thing also hold for the ESS? The answer is no, because the ESS is a stronger concept than the Nash equilibrium: every ESS corresponds to a Nash equilibrium, but the reverse is not true, because the definition of an ESS is more restrictive.

To see that not all Nash equilibria are ESSs, consider the Rock-Paper-Scissors game with the payoff matrix

	R	P	S
R	0 , 0	-1 , 1	1 , -1
P	1 , -1	0 , 0	-1 , 1
S	-1 , 1	1 , -1	0 , 0

We have seen that $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ corresponds to a Nash equilibrium. Is it also an ESS? The first ESS condition fails, and the second ESS condition fails too:

$$\pi_1(\mathrm{R}, \mathrm{R}) - \pi_1(x, \mathrm{R}) = 0$$

and so x is not an ESS.

However, for two-person games with two strategies we have the following result:

Proposition. Every two-person game with two pure strategies and payoff matrix

	x_1	x_2
x_1	a, a	b, c
x_2	c, b	d, d

has an ESS if mixed strategies are allowed and $a \neq c$ and $d \neq b$.

Proof under reconstruction.

Note that a similar result doesn't hold for more than two strategies as the above Rock-Paper-Scissors game demonstrated.

Proposition. In a symmetric 2-by-2 payoff matrix game, the ESS conditions are equivalent to

$$\pi_1(x, x') < \pi_1(x', x')$$

or

$$\pi_1(x, x') = \pi_1(x', x') \text{ and } \pi_1(x, x) < \pi_1(x', x)$$

for every pure strategy $x \neq x'$.

Proof is left as an exercise. Note that this proposition is quite practical: to check whether a strategy is an ESS, it is enough to test it against pure strategies!

The following is a handy little proposition that will save you the trouble of having to check for the existence of certain ESSs.

Proposition. If a game has two ESSs, then the support of the one cannot be a subset of the other.

Proof. Let x' and y' be two evolutionarily stable strategies. To reach a contradiction, suppose that the support of x' is a subset of the support of y'. Then, by the Bishop-Cannings theorem, $\pi_1(y', y') = \pi_1(y_i, y')$ for all y_i in the support of y', and hence for all y_i in the support of x'. Then $\pi_1(x', y') = \sum_i p_i \pi_1(y_i, y') = \pi_1(y', y')$, where p_i is the probability that the pure strategy y_i in the support of x' is played. Thus, as y' is an ESS (and since the first ESS condition fails) the second ESS condition should hold, in particular, $\pi_1(x', x') < \pi_1(y', x')$. But this contradicts that x', too, is an ESS. □

Remarks:

(1) If a game has a pure strategy ESS, then there cannot exist mixed strategy ESSs containing that pure strategy in their support. For example, in the PD game, D is an ESS and hence there cannot be a mixed ESS as well.

(2) Whenever in a given game there is an ESS with full support, then that ESS is the only one possible for that game. For example, the modified Rock-Paper-Scissors game above has an ESS with full support, which necessarily is the only ESS of the game.

(3) However, a game can have two (or more) mixed ESSs with non-empty intersection of their support. For example (see Haigh 1975), in a payoff matrix

	x_1	x_2	x_3
x_1	5 , 5	7 , 8	2 , 1
x_2	8 , 7	6 , 6	5 , 8
x_3	1 , 2	8 , 5	4 , 4

Here both $x' = (\frac{1}{4}, \frac{3}{4}, 0)$ and $x^* = (0, \frac{1}{3}, \frac{2}{3})$ are ESSs. Neither support is a subset of the others support, but the intersection is non-empty.

Lecture 11-2-2013

2.6. **Playing the field.** As a sort of intermezzo, to get an idea of another of population dynamical embedding of the notion of ESS, we have a brief look at a kind of game situation called "playing the field" where individuals are not involved in pair-wise contest but instead interact with the population as a whole. Hamilton and May (1977) gave an example of plants which had to decide on the fraction of seeds dropped nearby. The remaining seeds are equipped with an umbrella so they can disperse further and hence experience less competition, however, the mortality is higher. What fraction should the plant use to maximize its payoff (i.e. fitness - the expected number of offspring). Notice that, indeed, plants are not in a pairwise contest, but play against the whole population.

The example we work out is that of the evolution of the sex ratio: this is the ratio of the expected numbers of sons and daughters produced per female. In most species this ratio is close to one, but why is this so?

Mammals have the XY sex-determination system which more or less predisposes them to a sex ratio close to one, but notable exceptions are know, e.g., in chimpanzees (Boesch & Boesch-Achermann (2000) Oxford University Press, page 86), both towards a lower and towards a higher sex ratio.

Other sex-determination systems are more flexible, e.g., temperature sex-determination (most prominently in reptiles) or change of sex as in sequential hermaphrodites, which is quite common in fish and snails.

The upshot is that the sex-determination system in itself provides insufficient explanation for sex ratios observed in nature. Here we attempt to give a simple evolutionary explanation.

Let n and n' denote the population densities of the resident and the invader strategies. As strategies, however, we do not take the sex ratios themselves but rather the proportions x and x' of sons among the offspring; the sex ratios then are $x/(1-x)$ and $x'/(1-x')$. We further write $N = n + n'$ for the total population density and $\varepsilon = n'/N$ for the fraction of invaders.

We assume that all females are mated. In this way the 'fitness' of a given sex ratio strategy only depends on the success of males in finding a mate. We also assume that the expected number of children per female is independent of the sex ratio strategy. As a measure of 'fitness' of the strategy x', we therefore take the number $g_1(x', x, \varepsilon)$ of grandchildren produced per female.

Let λ denote the expected number of children per female. The number of daughters of a female with strategy x' then is $(1-x')\lambda$. Each of these daughters produces another λ children, and so the number of grandchildren produced via the daughters is $(1-x')\lambda^2$.

The number of grandchildren produced via the sons is equal to the number of sons (i.e., $x'\lambda$) times the number of females available per son (i.e., the total number of females in the population divided by the total number of males) times the number of children per female (i.e., λ), which is

$$x'\lambda^2 \, \frac{(1-\varepsilon)(1-x)N\lambda + \varepsilon(1-x')N\lambda}{(1-\varepsilon)xN\lambda + \varepsilon x' N\lambda}$$

The total number $g_1(x', x; \varepsilon)$ of grandchildren produced via sons and daughters then is

$$g_1(x', x; \varepsilon) = (1-x')\lambda^2 + x'\lambda^2 \, \frac{(1-\varepsilon)(1-x) + \varepsilon(1-x')}{(1-\varepsilon)x + \varepsilon x'}$$

Likewise, for the number of grandchildren produced per female with strategy x we find The total number $g_1(x', x; \varepsilon)$ of grandchildren produced via sons and daughters then is

$$g_2(x', x; \varepsilon) = (1-x)\lambda^2 + x\lambda^2 \, \frac{(1-\varepsilon)(1-x) + \varepsilon(1-x')}{(1-\varepsilon)x + \varepsilon x'}$$

Embedding the ESS definition into the present context gives that a sex ratio strategy x is evolutionarily stable if, and only if for every $x' \neq x$ there exists an $\varepsilon_0 > 0$ such that $g_1(x', x; \varepsilon) < g_2(x', x; \varepsilon)$ whenever $\varepsilon < \varepsilon_0$.

Expanding into terms of different order in ε, we get

$$g_1(x', x; \varepsilon) - g_2(x, x; \varepsilon) = -\lambda^2\Big(1 - 2x + 2x' - \frac{x'}{x}\Big) - \lambda^2 \frac{(x-x')^2}{x^2}\varepsilon + \mathrm{O}(\varepsilon^2)$$

It follows that x is evolutionarily stable if for every $x' \neq x$

$$1 - 2x + 2x' - \frac{x'}{x} > 0$$

or

$$1 - 2x + 2x' - \frac{x'}{x} = 0 \quad \& \quad \frac{(x-x')^2}{x^2} > 0$$

Only $x = 1/2$ satisfies the conditions, i.e., the first condition fails, but the second condition is satisfied. Conclusion: $x = 1/2$ (which corresponds to a sex ration of one) is an ESS.

Lecture 13-2-2013

3. Games with an infinite strategic form

Let us first look at an example.

In the Hawk-Dove game, two Doves share the resource. It may happen, however, that the resource is indivisible or simply not worth to divide (e.g. territory, mate). If this is the case, one would expect some kind of contest to emerge.

Such a contest without escalation (i.e., so that no-one gets injured) is called a 'display contest'. A display contest may in fact look more intimidating than an escalated fight, precisely because each contestant tries to intimidate the other by roaring, fake charges, stamping and jumping and whatever, and moreover may last longer than a real fight. In short, a display contest can be costly in terms of both time and energy, and the question is how much the contestants are ready to pay?

The payoff for obtaining the resource is R. The cost of displaying is assumed to be an increasing function of time and is denoted by c. If two players chose to invest a maximum cost of c_1 and c_2, respectively, then the one with the higher value will win, but he does not have to pay that cost, because the length of the contest is determined by the lower value. After all, there is no point in continuing the display if your opponent has already given up. If both players happen to choose the same cost, then the contest is decided randomly. This is called the *War of Attrition*. The payoffs thus are:

	Player 1	Player 2
$c_1 > c_2$	$R - c_2$	$-c_2$
$c_1 = c_2$	$R/2 - c_2$	$R/2 - c_2$
$c_1 < c_2$	$-c_1$	$R - c_1$

Remarks:

(1) Note that the choice of c is made before the contest, but that the cost is being paid only as the contests proceeds. That is why the actual cost being paid depends on the length of the game and thus on the minimum of the two choices.

(2) Also note that as the choice of c is made before the contest, there is no exchange of information about how long the opponent is going to continue, and so the value of c is not updated. For example, if one player is about

to give up, but he sees that his opponent is getting tired, then in reality he may change his mind and decide not to give up yet, because victory is in sight. This does not happen in the War of Attrition.

(3) As opposed to the assumptions in the War of Attrition, in a so-called *Size Game* the players invest resources in 'weaponry' such as body size, the size of antlers or horns, so that the cost is being paid before the actual contest but without 'refund' to the winner.

It is clear that there is no pure strategy that is an ESS: the higher value of c always wins and can invade. Curiously enough, this even holds if the costs exceed the value of the resource! If there is an ESS it must be a mixed strategy.

Let $F : [0, \infty) \to [0, 1]$ be the cumulative distribution function of a mixed strategy, and suppose that F is everywhere continuously differentiable. Its derivative $f = F'$ is the probability density function, and

$$F(c) = \int_0^c f(\gamma)d\gamma$$

is the probability of choosing a cost less than or equal to c. The payoff to a pure strategy c against F is

$$\pi_1(c, F) = \int_0^c (R - \gamma)f(\gamma)d\gamma - c\int_c^\infty f(\gamma)d\gamma$$

If F is evolutionarily stable, then by the Bishop-Cannings theorem (a generalized version) we have

$$\pi_1(c, F) = \pi_1(F, F).$$

Differentiation with respect to c gives

$$(R - c)f(c) - \int_c^\infty f(\gamma)d\gamma + cf(c) = 0$$

which we rewrite as

$$F'(c) = \frac{1}{R}\Big(1 - F(c)\Big)$$

Solving this differential equation for the initial condition $F(0) = 0$ gives

$$F(c) = 1 - e^{-c/R}$$

with corresponding probability density function

$$f(c) = \frac{1}{R}\, e^{-c/R}$$

which belongs to the so-called exponential distribution with expectation R.

Conclusion: if there is a mixed ESS with a continuous probability density function, then this must be the exponential distribution.

Proposition. The exponential distribution with the cumulative distribution function

$$F(c) = 1 - e^{-c/R}$$

is the unique ESS of the War of Attrition.

Proof. Since the support of one ESS cannot be contained in the support of another ESS, and F has full support, it is sufficient to show that F is an ESS; uniqueness is implied automatically.

We have $\pi_1(c, F) = \pi_1(F, F)$ for all $c \geq 0$; that is how we constructed F in the first place. Hence, for any arbitrary distribution function G on $[0, \infty]$ we have

$$\pi_1(G, F) = \int_0^\infty \pi_1(c, F) dG(c) = \int_0^\infty \pi_1(F, F) dG(c)$$

$$= \pi_1(F, F) \int_0^\infty dG(c) = \pi_1(F, F)$$

so the first ESS condition fails, and we have to check the second ESS condition.

For two arbitrary distribution functions G_1 and G_2, let $H_{G_1 \times G_2}$ be the cumulative probability distribution of the actual cost of a $G_1 \times G_2$-contest, i.e.,

$$\begin{aligned}
H_{G_1 \times G_2}(c) &= \text{Prob}\left\{ \begin{array}{l} \text{at least one of the players gives up before or at} \\ \text{the moment the cost has reached the value } c \end{array} \right\} \\
&= 1 - \text{Prob}\left\{ \begin{array}{l} \text{neither player gives up before or at the} \\ \text{moment the cost has reached the value } c \end{array} \right\} \\
&= 1 - \Big(1 - G_1(c)\Big)\Big(1 - G_1(c)\Big) \\
&= G_1(c) + G_2(c) - G_1(c)G_2(c)
\end{aligned}$$

For the total payoff, $\pi_1(G_1, G_2) + \pi_1(G_2, G_1)$, we then have

$$\begin{aligned}
\pi_1(G_1, G_2) + \pi_1(G_2, G_1) &= \int_0^\infty (R - 2c) dH_{G_1 \times G_2}(c) \\
&= R - 2 \int_0^\infty c \, d\Big(G_1(c) + G_2(c) - G_1(c)G_2(c)\Big)
\end{aligned}$$

For $G_1 = G_2 = F$ we thus have

$$\pi_1(F, F) = \tfrac{1}{2} R - \int_0^\infty c \, d\Big(2F(c) - F(c)^2\Big) \tag{3}$$

$$= \tfrac{1}{2} R - 2 \int_0^\infty c(1 - F(c)) \, d(F(c)) \tag{4}$$

and for $G_1 = G_2 = G$ we have

$$\pi_1(G,G) = \tfrac{1}{2}R - \int_0^\infty c\, d\Big(2G(c) - G(c)^2\Big) \tag{5}$$

$$= \tfrac{1}{2}R - 2\int_0^\infty c(1-G(c))\, d(G(c)) \tag{6}$$

and for $G_1 = F$ and $G_2 = G$ we have

$$\pi_1(F,G) + \pi_1(G,F) = R - 2\int_0^\infty c\, d\Big(F(c) + G(c) - F(c)G(c)\Big)$$

Consequently (using the first ESS condition and integrating by parts),

$$\begin{aligned}
\pi_1(G,G) - \pi_1(F,G) &= \pi_1(G,G) - \pi_1(F,G) - \pi_1(G,F) + \pi_1(F,F) \\
&= -2\int_0^\infty c(1-F(c))dF(c) - 2\int_0^\infty c(1-G(c))dG(c) \\
&\quad +2\int_0^\infty cd(F(c)+G(c)-F(c)G(c)) \\
&= 2\int_0^\infty c(F(c)-G(c))d(F(c)-G(c)) \\
&= 2[c(F(c)-G(c))^2]_0^\infty - 2\int_0^\infty (F(c)-G(c))^2 dc \\
&\quad -2\int_0^\infty c(F(c)-G(c))d(F(c)-G(c))
\end{aligned}$$

$$\Longleftrightarrow$$

$$2\int_0^\infty c\, d\Big(F(c)-G(c)\Big)^2 dF(c) = [c(F(c)-G(c))^2]_0^\infty - \int_0^\infty (F(c)-G(c))^2 dc.$$

Since $[c(F(c)-G(c))^2]_0^\infty = 0$, we get

$$\begin{aligned}
2\int_0^\infty c\, d\Big(F(c)-G(c)\Big)^2 dF(c) &= \pi_1(G,G) - \pi_1(F,G) \\
&= -\int_0^\infty (F(c)-G(c))^2 dc \\
&< 0 \quad \forall G \neq F
\end{aligned}$$

which means that the second ESS condition is satisfies, and so F is indeed an ESS.
□

Lecture 18-2-2013

Before we analyze the Hawk-Dove by applying the War of Attrition, let us formalize some concepts discussed in the previous lecture.

Definition A game is said to have an *infinite strategic form* when the strategy sets for each player consist of a continuum of (pure) strategies.

Such continuous strategies are for example body size, skin/fur color, waiting time, investment of energy, level of aggression etc. All of these pure strategies that characterize an individual or its behavior are taken from a continuum of options. (Also discrete strategies, such as monetary investment, is often more useful to think as a continuous strategy.) What is nice, that many above results which were derived for games with a finite strategic form can be generalized to games with an infinite strategic form. The following propositions are perhaps the most useful.

Let F and G be two different distribution functions on a given interval $X \subset R$, and suppose that F, G and the payoff function π are continuous with respect to the strategies in X.

Proposition (Generalized Bishop-Cannings theorem) If F is an ESS, then $\pi_1(x, F) = \pi_1(F, F)$ for all $x \in X$ in the support of F.

Proposition If F and G are both ESSs, then the support of one cannot be a subset of the other (and *vice versa*).

The proofs of these propositions are left as an exercise.

3.0.1. *The War of Attrition in the Hawk-Dove game.* In the exercises, we calculated the ESSs of the original HD-game

	H	D
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	R , 0
D	0 , R	$\frac{1}{2}R$, $\frac{1}{2}R$

- If $R \geq C$: H is a unique ESS.
- If $R < C$: a mixed strategy $x = (\frac{R}{C}, 1 - \frac{R}{C})$, where $\frac{R}{C}$ is the probability to play Hawk, is a unique ESS.

Supposing that the population is at an ESS, the predicted frequency of escalated fights in the population (i.e., the frequency of H×H-contests) is therefore one if

$R \geq C$, and R^2/C^2 if $R < C$.

Now, suppose that the Dove-Dove encounter is as in the War of Attrition, where the players are engaged in a display contest. The ESS in the WA is a mixed strategy with a cumulative distribution function $F(c) = 1 - e^{-\frac{c}{R}}$ (see above). Supposing that Doves have evolved this strategy, the expected payoff in a D vs. D (or F vs. F) encounter is a

$$\pi_1(F, F) = \pi_1(0, F) = \int_0^0 (R - \gamma) f(\gamma) d\gamma - 0 \cdot \int_0^\infty f(\gamma) d\gamma = 0$$

In other words, against an opponent who plays the strategy F, there is no expected gain from trying to obtain the resource.

The modified Hawk-Dove game in which we re-calculated the payoffs for the display contest (i.e., between Doves) has the payoff matrix

	H	D
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	R , 0
D	0 , R	0 , 0

where

- If $R \geq C$: H is a unique ESS.
- If $R < C$: a mixed strategy $x = (\frac{2R}{C+R}, 1 - \frac{2R}{C+R})$, where $\frac{2R}{C+R}$ is the probability to play Hawk, is a unique ESS.

In the latter case, the predicted frequency of escalated fights is even higher than in the original Hawk-Dove game. Yet, escalated fights, although they do occur, are not at all *that* common in reality as suggested by the results above. To explain why, we introduce the notion of an asymmetric game.

4. Asymmetric games

Until now, the evolutionary games we considered were symmetric in the sense that the players were interchangeable: they had the same strategy set and the same payoffs. This is often and maybe even typically not the case: one player may be stronger than the other; the value of the resource or the expected costs may be different for the two players; in some games one player is male and the other female, and so forth. Such asymmetries may affect the payoffs as well as the strategy sets in a way that is different for the two players. What we need is an ESS solution concept that applies to such asymmetries.

The idea is to use so-called 'conditional strategies', which allow us to reformulate an asymmetric game as a symmetric game for which we already have an ESS solution concept, and then translate the result back to the asymmetric situation.

This is how it works: suppose that

(1) every contest is between a pair of individuals one of which is in role X (e.g., 'larger', 'older', 'stronger') and the other in role Y (e.g., 'smaller', 'younger', 'weaker')

(2) both players know for sure which role they occupy

(3) both players have the same set of conditional strategies from which they can choose

A conditional strategy is of the form $(x, y) \in X \times Y$, meaning "if in role X, play strategy $x \in X$; if in role Y, play strategy $y \in Y$". The sets X and Y need not be the same. Note that the players will end up in different roles, but they possess the same strategies to choose from! In other words, the game has become symmetric again, and so we can apply the same old notion of evolutionary stability to conditional strategies.

Let $z = (x, y)$ and $z' = (x', y')$ be two conditional strategies. The expected payoff to z against z' is

$$E(z, z') \stackrel{\text{def}}{=} \frac{1}{2}\pi_1(x, y') + \frac{1}{2}\pi_2(x', y)$$

Applying the definition of an ESS, we have that z' is an ESS if for every $z \neq z'$ either

$$E(z, z') < E(z', z')$$

or

$$E(z, z') = E(z', z') \quad \& \quad E(z, z) < E(z', z)$$

Writing out these conditions for the payoff functions π_1 and π_2 we get:

Definition. The conditional strategy $(x', y') \in X \times Y$ is an ESS if for all $(x, y) \neq (x', y')$ either

$$\pi_1(x, y') + \pi_2(x', y) < \pi_1(x', y') + \pi_2(x', y')$$

or

$$\begin{aligned} &\pi_1(x, y') + \pi_2(x', y) = \pi_1(x', y') + \pi_2(x', y') \\ \& \quad &\\ &\pi_1(x, y) + \pi_2(x, y) < \pi_1(x', y) + \pi_2(x, y') \end{aligned}$$

The following proposition and its corollary are actually quite neat:

Proposition. The conditional strategy $(x', y') \in X \times Y$ is an ESS if and only if it is a *strict* Nash equilibrium.

Proof. By definition, (x', y') is a *strict* Nash equilibrium if

$$\begin{aligned} \pi_1(x, y') < \pi_1(x', y') \quad \forall x \neq x' \\ \pi_2(x', y) < \pi_2(x', y') \quad \forall y \neq y' \end{aligned}$$

To prove the implication in one direction, suppose (x', y') is a strict Nash equilibrium, then

$$\pi_1(x, y') + \pi_2(x', y) < \pi_1(x', y') + \pi_2(x', y')$$

whenever $x \neq x'$ or $y \neq y'$, i.e., $(x, y) \neq (x', y')$, and so (x', y') is an ESS.

To prove the implication in the other direction, suppose that (x', y') is an ESS. First we show that (x', y') is a Nash equilibrium. From the definition we know that

$$\pi_1(x, y') + \pi_2(x', y) \leq \pi_1(x', y') + \pi_2(x', y') \quad \forall (x, y)$$

For $y = y'$ this gives

$$\pi_1(x, y') \leq \pi_1(x', y') \quad \forall x$$

and likewise, for $x' = x$,

$$\pi_2(x', y) \leq \pi_2(x', y') \quad \forall y$$

which shows that (x', y') is a Nash equilibrium.

Now we show that (x', y') is a *strict* Nash equilibrium. To reach a contradiction, suppose that this were not true. Then

$$\exists\, x'' \neq x' : \quad \pi_1(x'', y') = \pi_1(x', y')$$

or

$$\exists\, y'' \neq y' : \quad \pi_2(x', y'') = \pi_2(x', y')$$

Without loss of generality we assume the latter. Then for $(x, y) = (x', y'')$ we have

$$\pi_1(x, y') + \pi_2(x', y) = \pi_1(x', y') + \pi_2(x', y'') = \pi_1(x', y') + \pi_2(x', y')$$

and so the first ESS condition fails. Since by assumption (x', y') is an ESS, the second ESS condition then must hold, i.e.,

$$\pi_1(x, y) + \pi_2(x, y) < \pi_1(x', y) + \pi_2(x, y')$$

which for $(x, y) = (x', y'')$ becomes

$$\pi_1(x', y'') + \pi_2(x', y'') < \pi_1(x', y'') + \pi_2(x', y') = \pi_1(x', y'') + \pi_2(x', y'')$$

and so

$$\pi_2(x', y'') < \pi_2(x', y'')$$

which is a contradiction that proves that (x', y') is in fact a *strict* Nash equilibrium. This completes the proof of the proposition. □

Corollary. An evolutionarily stable conditional strategy $(x, y) \in X \times Y$ always consists of pure strategies.

Proof. A Nash equilibrium in which one of the strategies is a mixed strategy is, by the Bishop-Cannings theorem, not a *strict* Nash equilibrium. □

Remarks: (1) The above corollary is a very powerful result, because finding a candidate mixed ESS and proving that it is an ESS indeed, is not always trivial for games with more than two pure strategies. However, in asymmetric games the ESS *never* is a mixed ESS. (2) Moreover, the nature of the asymmetry nor the degree of asymmetry do not matter at all as long as both players recognize the difference. The assignment of roles is more like a pre-game contract than a signal: there may be no correlation between the asymmetry and the actual strength of the contestants, for example.

Lecture 20-2-2013

We will go through three examples, which will hopefully demonstrate the potential, and perhaps also some weaknesses, of asymmetric games.

Example 1. Consider the Hawk-Dove game where the resource is a territory and where the roles are defined as territory 'owner' (row-player) and 'intruder' (column-player). The strategy set consists of four conditional strategies: {(H,H), (H,D), (D,H), (D,D)} where the first of each pair is adopted in the role of 'owner' and the second in the role of 'intruder'.

	H	D
H	$\frac{1}{2}(R-C)$, $\frac{1}{2}(R-C)$	R , 0
D	0 , R	$\frac{1}{2}R$, $\frac{1}{2}R$

From the payoff matrix we immediately see that:

- If $R > C$, then (H,H) is a strict Nash equilibrium and therefore is an ESS.
- If $R = C$, then (H,H) is a Nash equilibrium but not a strict Nash equilibrium, and therefore is not an ESS. In fact, there is no ESS.
- If $R < C$, then (H,D) and (D,H) are both strict Nash equilibria, and therefore both are ESS.

The conditional strategy (H,D), which prescribes that 'if you're the owner of a territory, play Hawk; if you're an intruder, play Dove', is called the Bourgois strategy. The opposite, (D,H), which prescribes that 'if you're the owner of a territory, play Dove; if you're an intruder, play Hawk', is called the Paradoxical strategy, because it goes contrary to our instincts. Both the Bourgois and the Paradoxical strategies are evolutionarily stable whenever $R < C$.

As an example of the Paradoxical strategy, John Maynard Smith (1982) mentions the social spider *Oecibus civitas*: these spiders live in groups, but each has its own web and refuge hole. If a spider is driven from its refuge hole, it darts off and enters another web. The owner of that web then darts off and enters another web. So, an initial intrusion causes an avalanche of displacements. The Bourgois strategy, however, is by far the more common of the two ESSes.

Example 2. Let's assume that the owner of the territory has a probability p of winning an escalated fight against the intruded. For example, one could argue that the owner knows its territory well enough to give him an advantage over the intruder. If this is so, the value of p is closer to one. The payoff matrix then becomes (where the row player is the owner):

	H	D
H	$pR-(1-p)C$, $(1-p)R-pC$	R , 0
D	0 , R	$\frac{1}{2}R$, $\frac{1}{2}R$

We immediately see that:

- (H,H) is an ESS $\Longleftrightarrow pR > (1-p)C$ and $(1-p)R > pC$.
- (H,D) is an ESS $\Longleftrightarrow (1-p)R < pC$.
- (D,H) is an ESS $\Longleftrightarrow pR < (1-p)C$.
- (D,D) is never an ESS.

This is equivalent to:

- (H,H) is an ESS $\Longleftrightarrow C/(R+C) < p < R/(R+C)$.
- (H,D) is an ESS $\Longleftrightarrow p > R/(R+C)$.
- (D,H) is an ESS $\Longleftrightarrow p < C/(R+C)$.
- (D,D) is never an ESS.

Graphically, this can be represented by the following two diagrams:

```
                   (D,H)              (H,H)              (H,D)
Case R>C:    |---------------|---------------|---------------|
             0            C/(R+C)         R/(R+C)            1
                                     p -->
```

and

```
                  (D,H)           (D,H)&(H,D)            (H,D)
Case R<C:    |---------------|---------------|---------------|
             0            R/(R+C)         C/(R+C)            1
                                     p -->
```

It is immediately clear that for p sufficiently close to one, the strategy (H,D), i.e., ‘choose Hawk if you are the owner; choose Dove if you are intruder’, is the only ESS.

Remarks: (1) Here we see a very different solution compared to the prediction of high-frequency of escalated fights in the original HD-game and the HD-game with a display contest. Introducing an asymmetry were one of the ’Hawk’-players has an advantage over the other, shows that ESS is (H,D), i.e. no escalated fights at all!

(2) Now, lets go back to the example 1 and suppose that in the Hawk-Dove game the asymmetry is due to differences in strength rather than ownership. Then we find the same ESS (since the roles were just labels), including the Paradoxical strategy in which the stronger player immediately surrenders the resource to the weaker player without a fight: this is absurd! Example 2 solves this problem by assigning an arbitrary probability to win the H vs. H fight. Choosing $p > \frac{1}{2}$ gives the advantage to the stronger (row) player.

Differences in gender or ownership are usually quite clear to both players. Differences in strength, size or age, however, are may be less obvious and may require some pre-game assessment. This assessment may take the form of a display contest and may be quite costly. The question arrises whether it is worth the trouble to assign roles at all, or whether it is more profitable to play an unconditional strategy. The following model is not very good (for reasons given later), but it may give some idea how to tackle this question.

Example 3. Consider the strategies Hawk, Dove and Assessor. The Assessor chooses Hawk if in one role (whichever role this is) and Dove in the other role. There is a cost of assessment, however, which we denote by γ. We also assume that after assessment, the Assessor finds itself in either role with probability one-half.

Let $x = (p, 1-p)$ denote a mixed strategy for choosing Hawk or Dove, and let A denote the Assessor strategy. Then

$$\begin{aligned}
\pi_1(x, A) &= \tfrac{1}{2}\pi_1(x, H) + \tfrac{1}{2}\pi_1(x, D) \\
&= \tfrac{1}{2}\Big(p \cdot \tfrac{R-C}{2} + (1-p) \cdot 0\Big) + \tfrac{1}{2}\Big(p \cdot R + (1-p) \cdot \tfrac{R}{2}\Big) \\
&= \tfrac{R}{4} + \tfrac{1}{2}\Big(R - \tfrac{C}{2}\Big)p \\
\pi_1(A, A) &= \tfrac{1}{2}\pi_1(H, D) + \tfrac{1}{2}\pi_1(D, H) - \gamma \\
&= \tfrac{1}{2}R - \tfrac{1}{2} \cdot 0 - \gamma \\
&= \tfrac{1}{2}R - \gamma
\end{aligned}$$

The Assessor strategy is evolutionarily stable if and only if $\pi_1(x, A) < \pi_1(A, A)$ for all x, i.e.,

$$(*) \quad \frac{R}{4} + \frac{1}{2}\Big(R - \frac{C}{2}\Big)p < \frac{1}{2}R - \gamma \quad \forall x$$

We distinguish the following three cases:

Case: $0 < R \leq \frac{1}{2}C$
The left hand side of $(*)$ has a maximum for $p = 0$. Substitution of $p = 0$ into $(*)$ shows that the Assessor strategy is an ESS if and only if $0 < \gamma < R/4$.

Case: $\frac{1}{2}C < R < C$
The left hand side of $(*)$ has a maximum for $p = 1$. Substitution of $p = 1$ into $(*)$ shows that the Assessor strategy is an ESS if and only if $0 < \gamma < (C - R)/4$.

Case: $C < R$
The left hand side of $(*)$ has a maximum for $p = 1$, but substitution of $p = 1$ into $(*)$ requires that $\gamma < 0$, which cannot be satisfied. Instead, the pure strategy Hawk is evolutionarily stable.

We conclude that assessing asymmetry in the Hawk-Dove game and playing the conditional strategy (H,D) is evolutionarily stable provided the cost of the assessment is not too high and the value of the resource does not exceed the cost of injury.

The problem with the above model is the probability of finding oneself in either role is hardly ever one-half. Consider for example asymmetry in strength: if the Assessor is one of the stronger individuals in the population, then the probability of meeting a weaker opponent is definitely greater than one-half. The probability of having a stronger or weaker opponent depends on the Assessor's rank in the population as a whole.

This is easily corrected in the model if we assume that the Assessor knows his own rank. The interesting thing is that whether the Assessor strategy is evolutionarily stable, or not, now also depends on the Assessor's rank in the population as a whole. To work this out is left as an exercise.

Lecture 27-2-2013

In this lecture, we recapitulated some definitions and results from the first half of the course and then we played a 'Public Good'-game.

Lecture 11-3-2013

5. Public Good -game

As an interlude, let us describe a 'Public Good'-game. Typical examples of PG-games are group hunting, preserving common property, collaborations to ensure security (e.g. national defense), fresh air or even knowledge. Such games display a dilemma: defectors do better than co-operators. To see why, let us consider a well-studied experimental scenario. Suppose that six anonymous players are given 10 dollars each, and are asked to invest any portion of it to a common pool. The players know that the sum of ALL contributions will be tripled and divided equally amongst ALL the players - irrespective of the amount they invested.
If all players invest their 10 dollars, they all receive $60 \times 3/6 = 30$. But, if one player invests nothing, then each will receive 25, hence the players who invested money will have 25 on their account but the defector will have $10 + 25 = 35$! Thus selfish players ought to defect. The problem is that if ALL defect, then no-one will increase their stock and all what they will have is those 10 dollars.
In real experiments (e.g. by Fehr and Gächter 2001), in a one-shot game most players invest about half their initial amount. However, if the game goes on for a few rounds, the contributions decline from round to round, and will end up close to zero. Those players which contributed more than others will feel exploited and in the next round invest less; hence in the next round the other cooperators will feel exploited and will invest less in the following round. Cooperation will fail.

In human societies, we have set up institutions that punish such free riders (think of for example fining a person who litters parks or rides free on public transportation). It turns out that in the absence of such institutions, individuals take the law in their own hands. Indeed, in another experiment, players are given the opportunity to punish defectors. If someone feels that another player invested too little, three dollars were deducted from their account. This fine is collected by the experimenter, so it doesn't end up in the common pot. In fact, the player who punished must pay one dollar to enforce the punishment: this can be viewed as an energy or time expenditure to execute the punishment. It turns out, that in this game the average contributions are higher than without a punishment, and if the game is played repeatedly, the level of investment may increase to almost 10 dollars. The threat of punishment encourages people to be more cooperative! This may be an expected outcome, since the players are afraid to be punished in the

next round if they invested too little to the common pool. Remarkably, however, even if the players are shuffled such that no round is played with the same players (and hence exclude the possibility that the players who you punished will increase their investment next round and therefore be beneficial for you), this result will nevertheless hold: the threat of punishment increases the level of cooperation.
Similar experiments have been done by scientist from different fields (Sigmund 2007). For example, it was shown that if the game was masked as a social event rather than an economic one, players were more cooperative. Or, when the experimenter touched the player before each round was played, they tended to invest more. This might be a similar effect as in the experiment where in the university coffee room a picture of an eye is put above a donation-basket and people were asked to contribute to the costs of coffee and milk: the investments were on average almost 3 times higher than in the presence of a picture of a flower (Bateson et al. 2006).

Now, the above experiment extends the games we have studied (so far) in two ways. Firstly, it is an N-person game, and secondly, the game is repeated several times. In this and the following few lectures we will consider the latter extension.

6. Iterated games

Remember the Prisoner's Dilemma (PD) with the strategies 'defect' (D) and 'cooperate' (C) and payoff matrix

PD	D	C
D	P , P	T , S
C	S , T	R , R

(payoffs to the row-player) with $S < P < R < T$. 'Defect' is a strictly dominating strategy, and therefore (D,D) is the dominating strategy solution (as well as a unique Nash equilibrium and a unique ESS), even though *both* players would have a higher payoff if they played (C,C). But what if this game is repeated? Will the outcome be the same? This is the main question we will try to answer in the following few lectures.

Definition In an iterated game (or a repeated game or a supergame), the same game (called a stage-game) is played some number of times. Each repetition is called a 'round'. The number of rounds may be finite, infinite or random.

Le us denote the iterated prisoners dilemma with IPD.

Result If the IPD is played exactly N times and both players know this, then the strategy 'always defect' is (strictly) dominant.

The proof is by backward induction: 'defect' is (strictly) dominant in the last round, and so both players will defect in round N (as there are no more rounds and what is past is past, the round N can be viewed in isolation). As the round N can be taken for granted we need to consider only the first $N-1$ rounds. But for the $N-1$th round the same argument prescribes the same move and hence players should never cooperate.

Remark: The same applies if the game length is unknown but has a known upper limit (check!).

Corollary For cooperation to emerge between two players, the total number of rounds N must be random and unknown to the players.

6.1. **Random number of rounds.** The most common way to implement this into the model is by assuming that after each round there is a constant probability $\delta \in (0,1)$ that there is another round. (Alternatively, for example, we could assume that the number of rounds is given by a Poisson distribution). The parameter δ is sometimes called a discounting parameter.

The probability the game is iterated at least n times is δ^n, thus the probability that the game has exactly $n+1$ rounds is $\delta^n(1-\delta)$. The expected value of the *total number of rounds* M is

$$\begin{aligned} \mathrm{E}[M] &= \textstyle\sum_{n=1}^{\infty} n\mathrm{P}\{M=n\} \\ &= (1-\delta)\textstyle\sum_{n=1}^{\infty} n\delta^{n-1} \\ &= (1-\delta)\frac{d}{d\delta}\left(\textstyle\sum_{n=1}^{\infty}\delta^n\right) \\ &= (1-\delta)\frac{d}{d\delta}\left(\frac{1}{1-\delta}-1\right) \\ &= \frac{1}{1-\delta} \end{aligned}$$

(The total number of rounds M is a random variable with a geometric probability distribution.)

Let us denote by $A(n)$ the payoff in the nth round. The expected value of the *total payoff* is given by

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \delta^n (1-\delta)[A(0) + \cdots + A(n)] \\
&= (1-\delta)A(0) + \delta(1-\delta)[A(0) + A(1)] + \ldots \\
&= A(0) + \delta A(1) + \delta A(2) + \ldots \\
&= \sum_{n=0}^{\infty} \delta^n A(n).
\end{aligned} \tag{7}
$$

If $A(n)$ is uniformly bounded for all n, then (7) always converges to some value $A(\delta)$, for $0 \leq \delta < 1$. For example in IPD $A(n) \in \{S, P, R, T\}$ for all n, and hence the limit $A(\delta)$ exists for $0 \leq \delta < 1$.

The average *payoff per round* is given by

$$
\frac{A(\delta)}{1/(1-\delta)} = (1-\delta)A(\delta) = (1-\delta)\sum_{n=0}^{\infty} \delta^n A(n) \tag{8}
$$

Remarks: In the special case $\delta = 1$ (i.e. iterated game with infinitely many rounds) the total payoff $\sum_{n=0}^{\infty} A(n)$ may diverge. In this case it is convenient to consider the average (over time) payoff per round, i.e.

$$
\lim_{n \to \infty} \frac{A(0) + \cdots + A(n)}{n+1} \tag{9}
$$

provided it exists. If it does, the expression (9) is given by the limit of equation (8), i.e. by

$$
\lim_{\delta \to 1} (1-\delta)A(\delta) \tag{10}
$$

(Sigmund 2010).

Lecture 13-3-2013

6.2. **Strategies in iterated games.** A more mathematical formulation of definitions for iterated games will follow in the next lecture, at this moment, we will give a more heuristic treatment.

A strategy in an iterated game tells for each round what action to choose. Action is equivalent to the definition of strategies in one-shot games, e.g. in IPD action is D or C.

We can distinguish between

- fixed strategies
- random strategies
- rule-based strategies

Fixed strategies: a fixed sequence of actions to be played. Examples of fixed strategies in IPD:

- allD = (D,D,D,D,D,...)
- allC = (C,C,C,C,C,...)
- AltCD=(C,D,C,D,C,...)
- etc.

A random strategy is a sequence $(x_1, x_2, x_3, \dots)$ where $x_n \in [0, 1]$ is a mixed action (equivalent to a mixed strategy in one-shot games) in the n^{th} round. For example in IPD, $x_n = (p_n, 1 - p_n)$, where p_n is to play C in the nth round. Obviously, the fixed strategies form a subset of the random strategies.

In a rule-based strategy, the choice of action depends on the history of the game up to that moment. For example, take 'Tit for tat' (TFT) in IPD: "*Choose C in the first round; after that choose whatever your opponent did in the previous round.*" Successive rounds for TFT against an opponent thus look like:

TFT:		..	D	C	C	D	C	D	D	C	C	C	D	C	..
opponent:	..	D	C	C	D	C	D	D	C	C	C	D	C	..	

Tit for tat is an English saying meaning "equivalent retaliation": TFT rewards cooperation with cooperation and punishes defection with defection.

Another example of a rule-based strategy is 'Pavlov': "*Choose C in the first round; after that repeat the same action as in the previous round if your payoff was high* *(i.e., R or T); otherwise change.*" Successive rounds for Pavlov against the same opponent as above thus look like:

Pavlov:	..	C	D	D	D	C	C	D	C	C	C	C	D	D	..
opponent:	..	D	C	C	D	C	D	D	C	C	C	D	C	..	

Pavlov is a 'win-stay, lose-switch' strategy. Variations of Pavlov start with a defection, or recognize only T or all three P, R and T as high payoffs.

Let us define a strategy to be a memory-n strategy, if the player needs to remember the nth move back. Then, TFT and Pavlov are memory-1 strategies. An example of a memory-2 strategy (which remembers a move from two rounds ago) is 'Tit for two tats' (TFTT): "*Start with two rounds of C; after that respond with defecting only if the opponent defects twice in a row.*"

TFTT:			..	C	C	D	C	C	D	C	C	C	C	C	..
opponent:	..	D	C	D	D	C	D	D	C	C	C	D	C	..	

In the TFT strategy, once the opponent defects, the TFT player immediately responds by defecting on the next move. If a C (with some low probability) is misinterpreted by the opponent as a D, then two TFT players may accidentally get stuck in a (C×D)-(D×C)-cycle (or even a (D×D)-cycle), resulting in a poor outcome for both players. A TFTT player, however, will let the first defection go unchallenged as a means to avoid the above trap. Only if the opponent defects twice in a row, the tit for two tats player will respond by defecting.

Another example of a memory-2 strategy is 'Two tits for Tat' (TTFT): "*Start with C; after that respond with two defections to every defection of the opponent.*"

TTFT:		..	D	D	C	D	D	D	D	D	C	C	D	D	..
opponent:	..	D	C	C	D	C	D	D	C	C	C	D	C	..	

The strategy 'Grim' starts with C but changes to D after the very first defection by its opponent and plays D from then onwards. Grim is an example of a memory-∞ strategy: it never forgets and never forgives.

6.3. How to calculate payoffs in iterated games?

Example 1. Let us calculate payoffs for the iterated IPD when TFT plays against a modified version of Pavlov; A Pavlov which starts the first round with defection. We call this slightly more grim variant **Pavlov***. So the question is: what are the entries of the following payoff matrix?

IPD	TFT	Pavlov*
TFT	?	?
Pavlov*	?	?

TFT×TFT gets immediately in a (C×C)-cycle. The overall expected payoff E_{CC} to TFT against TFT is given by

$$E_{\mathrm{CC}} = R + \delta E_{\mathrm{CC}}$$

and hence

$$E_{\mathrm{CC}} = \frac{R}{1-\delta} \tag{11}$$

where $\delta \in (0,1)$ is the probability of a next round.

TFT×Pavlov* gets into a (C×D)-(D×D)-(D×C)-cycle. Let now E_{CD} and E_{DD} and E_{DC} denote the overall payoffs to TFT if the cycle is started in (C×D) or (D×D) or (D×C), respectively. Then we have

$$\begin{cases} E_{\mathrm{CD}} = S + \delta E_{\mathrm{DD}} \\ E_{\mathrm{DD}} = P + \delta E_{\mathrm{DC}} \\ E_{\mathrm{DC}} = T + \delta E_{\mathrm{CD}} \end{cases}$$

which is readily solved for E_{CD}, E_{DD} and E_{DC}. Of these we only need the first one, because that's how a TFT×Pavlov* contest actually starts. This gives us

$$E_{\mathrm{CD}} = \frac{S + \delta P + \delta^2 T}{1-\delta^3} \tag{12}$$

Pavlov*×TFT gets into a (D×C)-(D×D)-(C×D)-cycle. If E_{DC} and E_{DD} and E_{CD} denote the overall payoffs to Pavlov* for different starting points in the cycle, we have

$$\begin{cases} E_{\mathrm{DC}} = T + \delta E_{\mathrm{DD}} \\ E_{\mathrm{DD}} = P + \delta E_{\mathrm{CD}} \\ E_{\mathrm{CD}} = S + \delta E_{\mathrm{DC}} \end{cases}$$

from which we solve

$$E_{\mathrm{DC}} = \frac{T + \delta P + \delta^2 S}{1-\delta^3} \tag{13}$$

Pavlov*×Pavlov* gives (D×D) in the first round followed by a (C×C)-cycle. With E_{DD} and E_{CC} denoting the overall payoffs to Pavlov* for different starting points, we have

$$\begin{cases} E_{\mathrm{DD}} = P + \delta E_{\mathrm{CC}} \\ E_{\mathrm{CC}} = R + \delta E_{\mathrm{CC}} \end{cases}$$

from which we solve

$$E_{\mathrm{DD}} = P + \frac{\delta R}{1-\delta} \tag{14}$$

For the overall game we collect the payoffs (11)-(14) in the payoff matrix:

IPD	TFT	Pavlov*
TFT	$\frac{R}{1-\delta}$, $\frac{R}{1-\delta}$	$\frac{S+\delta P+\delta^2 T}{1-\delta^3}$, $\frac{T+\delta P+\delta^2 S}{1-\delta^3}$
Pavlov*	$\frac{T+\delta P+\delta^2 S}{1-\delta^3}$, $\frac{S+\delta P+\delta^2 T}{1-\delta^3}$	$P+\frac{\delta R}{1-\delta}$, $P+\frac{\delta R}{1-\delta}$

We conclude that TFT is an ESS if

$$\frac{R}{1-\delta} > \frac{T+\delta P+\delta^2 S}{1-\delta^3}$$

and that Pavlov is an ESS if

$$P+\frac{\delta R}{1-\delta} > \frac{S+\delta P+\delta^2 T}{1-\delta^3}$$

One readily shows that TFT is an ESS for sufficiently large $\delta \in (0,1)$, i.e., if the number of rounds tends to be high. It is bit more difficult to see the full parameter region where Pavlov* is an ESS against TFT; but at least for small δ or high enough T.

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Example 2. We play allD (the champion of the IPD with a fixed number of rounds) against TFT.

TFT×TFT gets immediately in a (C×C)-cycle, and the overall expected payoff E_{CC} to TFT against TFT (as calculated previously) is

$$E_{\mathrm{CC}} = \frac{R}{1-\delta}$$

TFT×allD gives (C×D) in the first round followed by a (D×D)-cycle. The overall payoff when starting with (C×D) is calculated as illustrated in the previous section and turns out to be

$$E_{\mathrm{CD}} = S + \frac{\delta P}{1-\delta}$$

allD×TFT gives (D×C) in the first round followed by a (D×D)-cycle. The overall payoff when starting with (D×C) is

$$E_{\mathrm{DC}} = T + \frac{\delta P}{1-\delta}$$

allD×allD immediately settles down in a (D×D)-cycle, and so the overall payoff to allD against allD is

$$E_{\mathrm{DD}} = \frac{P}{1-\delta}$$

For the payoff matrix of the IPD with strategies TFT and allD we thus have

IPD	TFT	allD
TFT	$\frac{R}{1-\delta}$, $\frac{R}{1-\delta}$	$S + \frac{\delta P}{1-\delta}$, $T + \frac{\delta P}{1-\delta}$
allD	$T + \frac{\delta P}{1-\delta}$, $S + \frac{\delta P}{1-\delta}$	$\frac{P}{1-\delta}$, $\frac{P}{1-\delta}$

TFT is an ESS whenever $\frac{R}{1-\delta} > T + \frac{\delta P}{1-\delta}$, i.e., whenever $1 - \frac{R-P}{T-P} < \delta < 1$. In other words, TFT is an ESS against allD if δ is large and the game, on average, continues for many rounds. AllD is an ESS whenever $\frac{P}{1-\delta} > S + \frac{\delta P}{1-\delta}$, i.e., whenever $P > S$, which, by assumption, is always true.

Example 3. Normally, allC against allD is an all-time looser. But suppose that we equip allC with the rule "*Quit whenever you receive the sucker's payoff S*", and denote this new variant **allC***. Like TFT, allC* has a means of punishing a defector, not by reciprocaying the defection, but by simply quitting the game. An (allC*×allC*)-contest lasts an average of $(1-\delta)^{-1}$ rounds, as does a (allD×allD)-contest, but a (allC*×allD)-contest lasts only one round. The payoff matrix is

IPD	allC*	allD
allC*	$\frac{R}{1-\delta}$, $\frac{R}{1-\delta}$	S , T
allD	T , S	$\frac{P}{1-\delta}$, $\frac{P}{1-\delta}$

One readily sees that allD against allC* is always an ESS, and allC* is an ESS against allD whenever $\frac{R}{1-\delta} > T$, i.e., whenever $1 - \frac{R}{T} < \delta < 1$, i.e., whenever the average number of rounds is sufficiently high.

The moral of this all is that iterating a game may make a big difference.

Next, we set out to give a general description and some properties of iterated games. For this, we need to introduce several concepts.

6.4. **A more formal description of iterated games.** Our focal interest is to describe iterated games, so that is what we will do. However, a slightly more general version of what will follow describes all sorts of dynamical games. Therefore, the necessary extensions to obtain the general description will be pointed out in the Remarks.

Consider a symmetric, one-shot 2-player game characterized by a set of players $I = \{1, 2\}$, an action space A (equal for both players), and a payoff function

$$\pi : A \times A \to \mathbb{R}^2. \tag{15}$$

(For example, in an IPD the action space is just {C,D}). Using a discount factor δ (i.e. the probability to play the one-shot game again), a one-shot game is turned into an iterated one, which we will denote with $\Gamma(\delta)$.

Definition. A *history* at time (step) t is a list of actions played up to and including time $t-1$.

If $a_{t,i} \in A$ is the *action* played by player $i \in I$ at time t, then these histories are:

$$\begin{cases} h_1 = \emptyset \\ h_t = (a_1, \dots, a_{t-1}) = ((a_{1,1}, a_{1,2}), \dots, (a_{t-1,1}, a_{t-2,2})), \qquad t = 2, 3 \dots. \end{cases}$$

As there are no games before the first one, the history at round 1 is empty. We also use the following notation $a_t = (a_{t,1}, a_{t,2})$ for all t. The set of possible histories at t is

$$\begin{cases} H_1 = \{h_1\} \\ H_t = \prod_{i=1}^{t-1}(A \times A), \qquad t = 2, 3 \ldots. \end{cases}$$

The set of *all* possible histories is

$$H = \cup_{t=1}^{\infty} H_t.$$

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Definition. A *strategy* s_i for player $i \in I$ is a sequence of maps

$$s_i = (s_{t,i})_{t=1}^{\infty} = (s_{1,i}, s_{2,i} \dots)$$

where

$$s_{t,i} : H_t \to A$$

(i.e. it satisfies $s_{t,i}(h_t) \in A \ \forall h_t$).

Definition. *A strategy profile* s is a set of all strategies s_i where $i \in I$. A strategy profile includes exactly one strategy for every player.

For example, for $I = \{1, 2\}$ we write $s = (s_1, s_2)$, where s_i is the strategy of player i.

For two strategies s_1 and s_2 in a strategy profile s, the course of actions is determined by recursion

$$h_1^s = \emptyset$$

and

$$\begin{cases} a_t^s = (a_{t,1}^s, a_{t,2}^s) = (s_{t,1}(h_t^s), s_{t,2}(h_t^s)) \\ h_{t+1}^s = (a_1^s, \dots, a_t^s). \end{cases}$$

The sequence of actions $(a_1^s, a_2^s, \dots)$ is called the *path* (or the path of play) of the strategy profile $s = (s_1, s_2)$.

If S is a space of (pure) strategies for the iterated, symmetric, 2-person game, then we write

$$\tilde{\Pi}_i : S \times S \to \mathbb{R} \tag{16}$$

for the total payoff-function of a player $i \in I = \{1, 2\}$. The average discounted payoff (average payoff per round) we denote with $\Pi = (1 - \delta)\tilde{\Pi}$. From (8) we get that the average payoff per round for player s_1 against s_2 is

$$\Pi_1(s_1, s_2) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_1(a_t^s). \tag{17}$$

Remarks:

(1) Above description is readily generalizable to N-person symmetric iterated games by setting $I = \{1, \dots, N\}$.

(2) The probability to enter a new round may in general be time-dependent, in that case we write δ_t.

(3) Above we describe games that last potentially infinite number of rounds, hence we will sometimes denote them with Γ^{∞} as opposed to games that have a fixed number of rounds T, which we denote with Γ^{T}. In the latter case, of course, all the above summations, unions etc. will be only calculated for $t = 1, \dots, T$.

(4) With slight modifications, above notation can be generalized to so-called multi-stage games (see also the following lectures). In multi-stage games you may play each round a totally different game, as opposed to iterated games where you play always the same one-shot game. Therefore at each time t the action set may be different, i.e. in general $A_{t_1,i} \neq A_{t_2,i}$ for $t_1 \neq t_2$. Also, if the game is not symmetric, players may have different action sets, i.e. $A_{t,i} \neq A_{t,j}$ for $i \neq j$.

6.4.1. *Strategic and extensive forms - general theory.* In most parts of this course, we have been looking at games in their strategic (or normal) form. When considering one-shot games where all players make decisions, this form describes the full game. However, as we shall see shortly, some information might be lost when describing dynamic games (iterated, multi-stage) in the normal form. It is therefore necessary to describe the game in its extensive form.

Definition A game in *strategic (or normal) form* has the following elements

- the set of players $I = \{1, \dots, N\}$
- pure strategy space $X = X_1 \times \cdots \times X_N$
- payoff functions $\pi_i : X \to \mathbb{R}$ for players $i \in I$

This we couple with a solution concept.

Definition The *extensive form* of a game contains the following information

- the set of players $I = \{1, \dots, N\}$
- game tree (who move when; see below)
- set of payoffs
- what the choices are when the players move
- what each player knows when they make choices
- the probability distribution over any exogenous events (the effects of 'Nature' may change the game)

A *game tree* consists of

- number of decision/action nodes connected by branches

- branch connects head to tail node
- one root node and a set of terminal nodes
- a tree property: there must be exactly one path from a root node to any given terminal node

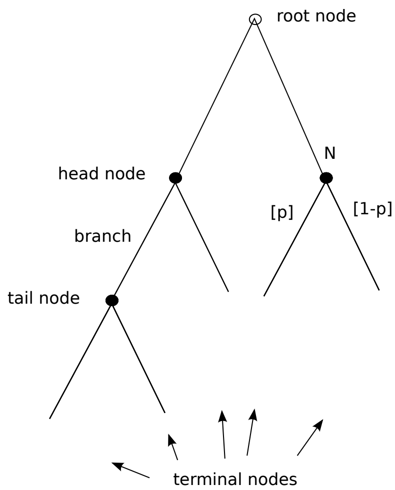

FIGURE 1. An example of a game-tree, where N (Nature) represents a stochastic event.

Remarks:

(1) We call (according to the definitions above)
   - a move at each node an action
   - series of actions that fully define behavior of a player a strategy

(2) a stochastic event may occur at a node x (e.g. bad or good weather may change the game): we assign a probability to each branch were x is the head node, representing the probability that 'Nature' chooses that branch (e.g. see Figure above).

Information sets model the information players have when they are choosing their actions.

Definition An *information set* for a player is a set of action nodes in a game tree such that

(1) the player concerned (and no other) is making decisions

(2) the player doesn't know which node has been reached

In other words, if at nodes x and x' the player has the same information, then x and x' belong to the same information set. In a game-tree the nodes that belong to the same information sets are connected with a dashed-line (see Example below).

Remaks:

- player must have the same choice at all nodes included in an information set
- a game has *perfect information* if all its information sets are singeltons

Example (Big Joe and Little Joe) This example was presented in the beginning of the course as well as in the exercises. In (a) the Big Joe makes his move first. Hence Little Joe, when its his time to move, knows what BJ did i.e. LJ knows at which node he is, either x or x'. In (b) BJ and LJ move simultaneously. As LJ doesn't know BJs move, LJ doesn't know whether he is at node x or x' and thus those nodes belong to the same information set. (Note that we could have alternatively drawn LJ to start at the root node.)

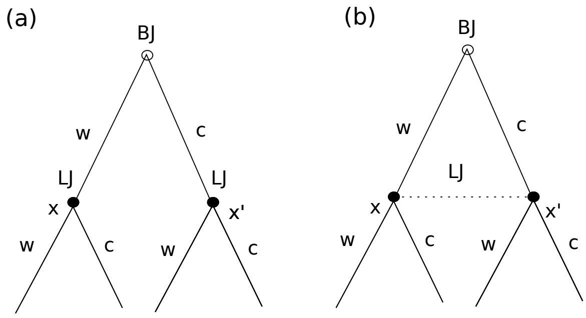

FIGURE 2. (a) BJ moves first (b) BJ and LJ move simultaneously.

Lecture 25-3-2013

Defintion Given any strategy profile s, an information set is said to be on the path of play if and only if the information set is reached with positive probability according to s. If s is a Nash equilibrium strategy profile then we refer to the *equilibrium path of play.*

Remark: If s is an ESS strategy profile we refer to the *ESS path of play.*

Example (Iterated Prisoners Dilemma $\Gamma(\delta)$): Suppose the strategy space is S = $\{allC, allD\}$ and consider a strategy profile $s = (s_1, s_2) = (allC, allD)$, i.e. there are two strategies to choose from and player 1 chooses $allC$ and player 2 chooses $allD$. We may write the strategies as $s_{t,1} = C$ and $s_{t,2} = D, \quad \forall t$. The path of play is

$$\begin{aligned}(a_1^s, a_2^s, a_3^s, \dots) &= \big((a_{1,1}^s, a_{1,2}^s), (a_{2,1}^s, a_{2,2}^s), (a_{3,1}^s, a_{3,2}^s), \dots\big) \\ &= ((s_{1,1}(\emptyset), s_{1,2}(\emptyset)), (s_{2,1}(h_2), s_{1,2}(h_2)), (s_{3,1}(h_3), s_{3,2}(h_3)), \dots) \\ &= ((C, D), (C, D), (C, D), \dots)\end{aligned}$$

See Figure 3. For s to generate an equilibrium path of play, i.e. for $(allC, allD)$

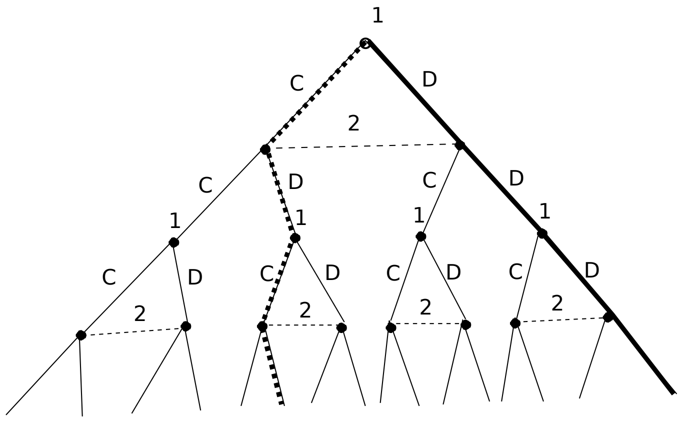

FIGURE 3. IPD with paths of play generated by two strategy profiles $s = (allC, allD)$ (thick dotted line) and $r = (allD, allD)$ (thick continuous line). The path generated by r is an equilibrium path of play.

to be a Nash equilbrium, we require that $\Pi_1(allC, allD) \geq \Pi_1(allD, allD)$ and $\Pi_2(allC, allD) \geq \Pi_2(allC, allC)$ (see Nash equilibrium conditions). As the first condition is not true $(allC, allD)$ is not a Nash equilibrium, and hence $((C, D), (C, D), (C, D), \dots)$ is not an equilibrium path of play.

However, a strategy profile $r = (r_1, r_2) = (allD, allD)$ is a Nash equilibrium and hence the path $((D, D), (D, D), (D, D), \dots)$ is an equilibrium path of play (see

Figure). As $allD$ is an ESS as well, the path is also an ESS path of play.□

What if we add some strategies to the strategy set, for example a TFT? We have an iterated prisoners dilemma with S = $(allC, allD, TFT)$. $allD$ is still a symmetric Nash equilibrium and an ESS, but now TFT is a symmetric NE and an ESS as well (depending on the parameters T, P, R, S and δ, see example 2 in Lecture 18.3. 2013).
Question is, can we make a general statement: are there strategies such as $allD$ or TFT that are symmetric Nash equilibria and ESSs if we allow the strategy set S to contain *any* strategies? The answer to the first question is yes.
To show this, we need to give two definitions.

Definition An iterated game $\Gamma(\delta)$ with a strategy set S is called *non-terminal* and *maximal* if no strategies in S terminate the game after a fixed number of rounds and if S contains all the possible strategies, respectively.

Sometimes we refer to the set S itself to be maximal and to the strategies to be non-terminal.

Theorem Consider a non-terminal and maximal iterated prisoners dilemma $\Gamma(\delta)$ with $T > R > P > S$ and $\delta > \max\{\frac{T-R}{T-P}, \frac{T-R}{R-S}\}$. Then, a strategy Tit For Tat is a symmetric Nash equilibrium.

Proof. (Axelrod and Hamilton 1981) Let us denote with s the strategy that gives against TFT the maximum payoff. As TFT remembers only one move back (i) one C by the other player in any round is sufficient to reset the situation as it was at the beginning of the game (ii) one D sets the situation to what it was at the second round after a D was played in the first. Now, for a strategy s to give the maximum payoff, the actions for both players at round x has to be followed by the same actions as if you play those actions at any other round x'. This is, because at each round the probability to play another round is always the same. It is then enough to look at the first two rounds (i) if s begins with C, then the second move must be C as well, as C was the best response to C in the first round and hence $s = allC$ which is a neutral mutant of TFT (ii) if s begins with D, then if (a) playing D at round 2 gives the maximum payoff then it must be the best to play it then onward, hence $s = allD$ (b) playing C at round 2 gives the best payoff then we get into $(C, D) \times (D, C)-$cycle, hence $s = altDC$.
We have then that if neither $allD$ nor $altDC$ can invade TFT then no other strategy can. It is straightforward to check that this is indeed the case for sufficiently large δ, more precisely, for $\delta > \max\{\frac{T-R}{T-P}, \frac{T-R}{R-S}\}$.□

Remarks: (i) In fact, Axelrod and Hamilton claimed that TFT is an ESS, however, their definition of an ESS was different (and wrong) (ii) $allD$ is a symmetric Nash equilibrium when *all* the strategies are allowed, even the ones that terminate the game after a fixed number of rounds (exercise). (iii) It can be argued that in an IPD with a non-terminal and maximal strategy set each strategy has infinitely many neutral mutants (exercise).

The answer to the second question, whether a non-terminal and maximal IPD has any ESSs, is no. In fact, we have a more general result: no non-terminal and maximal iterated games have an ESS.

Definition We call a strategy $r \neq s$ for which $\Pi_1(r,s) = \Pi_1(r,r) = \Pi_1(s,r) = \Pi_1(s,s)$ a *neutral mutant* of strategy s.

Theorem Every pure strategy in every non-terminal and maximal iterated game has neutral mutants.

Corollary There are no ESSs in non-terminal and maximal iterated games.

Proof of the Theorem. For every strategy s playing against itself there is always an off-equilibrium path (i.e. a path that is not taken when these strategies are played). On the off-equilibrium path a strategy can be changed without consequences to the payoffs. Changing strategy s on the off-equilibrium path creates a mutant r for which $\Pi_1(r,s) = \Pi_1(r,r) = \Pi_1(s,r) = \Pi_1(s,s)$. $\square$

The statement of the corollary follows directly, since an ESS can't allow for neutral mutants.

Remarks: To demonstrate the above result in the iterated prisoners dilemma let us look at the previous example where for $S = \{allC, allD\}$ an $allD$ was an ESS. We extend the strategy set by adding a strategy $r = allD'$, defined as "play D except if the opponent played C at round x then play C at round $x+1$". As this strategy is a neutral mutant of $allD$, strategy $allD$ is no more an ESS when the strategy set is $S' = \{allC, allD, allD'\}$. In any game, the maximal strategy set will contain such neutral mutants.

Lecture 27-3-2013

6.4.2. *Indirect invasions.* Recall that if a population is using an ESS, it means that no other strategies when rare can increase in frequency (i.e. invade) and hence replace the ESS strategy. An ESS thus can't be invaded by any rare strategy. But what if the strategy is "only" a (symmetric) Nash equilibrium? Can then rare strategies invade?
Invasion by a rare strategy was defined to be only possible if this rare strategy does strictly better in terms of payoffs when played against the common strategy used by the population than the common strategy against the common strategy (see the section where we defined an ESS). If this common strategy is a symmetric Nash equilibrium then by definition no other strategy can do strictly better and hence Nash equilibrium can't be invaded either.
However, a Nash equilibrium allows for neutral mutants, i.e. a Nash equilibrium is still a Nash equilibrium even if we extend the strategy set to contain its neutral mutants. Now, it is possible that a rare neutral mutant can increase in frequency due to *random* events. To demonstrate how random (or stochastic) events may cause the change of frequency is by considering random sampling with replacement: Take a box with 20 balls, 10 white and 10 red. Pick randomly one of the balls, check the color, make a copy of this ball and put it into a new box. Place the ball you picked up back into the old box, mix it and repeat the experiment 20 times. Obviously, counting the total number of red and white balls in the new box white and red balls may have increased or decreased in frequency. It may happen (in fact it happens with probability 1 if we repeat this experiment long enough) that all the balls will be only one color. Analogically we can then say, that even if a strategy is a Nash equilibrium it may have a neutral mutant which will increase in frequency due to stochastic events and take over the whole population such that it has completely replaced the initial Nash equilibrium strategy. Question is, why do we care? After all, the strategies (NE and its neutral mutant) behave the same against each other and as they do equally well they might as well could be considered as one and the same strategy. Well, this is not true: they might behave differently against a third strategy! It might be then possible, that a Nash equilibrium is wiped out by a neutral mutant and the neutral mutant is invaded by a third strategy. This type of invasion is called an *indirect invasion* and it deserves a formal definition.

Definition A strategy r is said to be able to invade a strategy s *indirectly* if it can invade a neutral mutant of strategy s.

Remark: This demonstrates the important difference of Nash equilibria and ESSs. ESSs can't be wiped out, Nash equilibrium can.

Proposition In iterated prisoners dilemma with $T > R > P > S$ a strategy TFT can be invaded indirectly if we allow for arbitrary strategies.

Proof. Strategy allC is a neutral mutant of TFT and allD can invade allC.

Remark: Bendor and Swistak (1996,1997,1998) showed that in IPD and for sufficiently large δ no symmetric Nash equilibrium is robust against indirect invasions.

6.4.3. *Subgame perfection - a general theory.* As mentioned above, a game that is transformed from an extensive form to a strategic form may loose some information. This was first pointed out in Selten (1965):

Example (Selten's game) Let us write the game tree in Figure 4

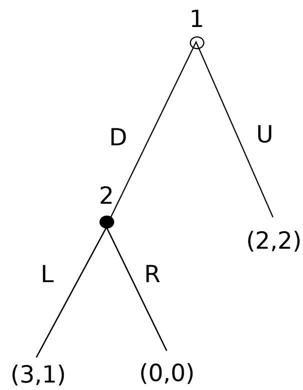

FIGURE 4. Game tree for the Selten's game

in its strategic form

	L	R
U	2 , 2	2 , 2
D	3 , 1	0 , 0

The game tree describes the game such, that if player 1 chooses U, then the game is finished. However, if the player 1 chooses D, then it is player 1's turn to make a choice between L and R (and only after this move the game ends). From the strategic form we find that the Nash equilibria (a necessary condition for it to be an ESS) are (D, L) and (U, R). However, there is something wrong with the latter one: it says that player 2 plays R, but, if it will ever be his turn to move (which is in the case player 1 played D) he would be obviously better off by choosing L (and receive a payoff 1 instead of 0)! The move R from player 2 is then a so-called *non-ncredible threat* (sometimes called an incredible threat), because he would never want to execute it. We thus conclude that the NE (U,R) is less plausible than (D,L) as it relies on a non-credible threat.

Next, we refine the concept of NE to be able to make a distinction between solutions of the above type.

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Definition A *subgame* is a part of a game-tree (sub-tree) that satisfies the following conditions

- it begins at a decision node
- the information set containing the initial decision node contains no other decision nodes
- the sub-tree contains all the decision that follow the initial node

Remarks Note that an entire tree is also a subgame.

Example In Figure 5 we show three game-trees. In the first one there are two subgames, the entire tree and one starting at node x. In the second we have three trees, the entire tree and subtrees starting at nodes x and y. In the last game-tree there is only one subgame, the entire tree. This is because a subtree must start in an information set with only one node.

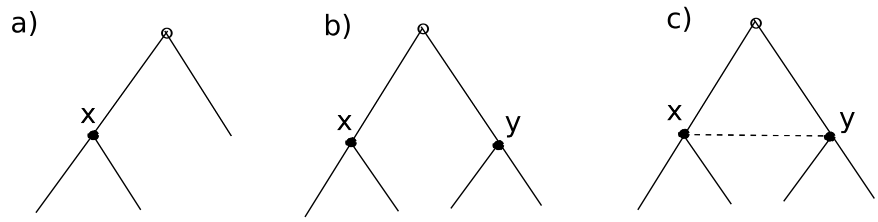


FIGURE 5.

Definition A subgame perfect Nash equilibrium (SPNE or SPE) is a Nash equilibrium of the whole game in which the behavior specified in every subgame is a Nash equilibrium for the subgame. This applies even to subgames that are not reached during a play of the game using Nash equilibrium strategies.

Example (A version of the 'Battle of the sexes' - sequential choice) Mike and John are deciding whether to go to the opera or to a football game. Mike likes opera (O) more and John likes football (F) more. However, they wouldn't enjoy either one if they don't go together. In this version, Mike chooses first.
The extensive form of the game is depicted in Figure 6 and the strategic/normal form is given below (John is the row player and Mike the column player). Recall that when moves are sequential, the strategy set for the second player consists of conditional strategies (the first letter tells what he will do if the first player chooses O and the second if the first player chooses F).

From the strategic form we see immediately that the pure strategy Nash equilibria are (O,OO), (F,FF) and (O,OF) (let us ignore mixed strategies in this example). Question is, which ones are also SPNE? The Nash equilibrium which prescribes

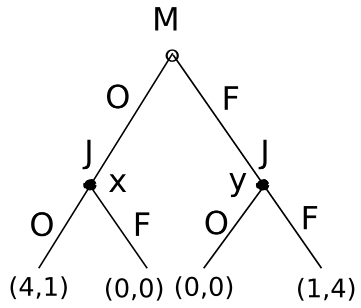


FIGURE 6.

a Nash equilibrium in each subgame is SPNE, and the only one which does so is the last one (O,OF). This is, because at node x (one of the three subgames) John should choose O (which is a NE for the subgame starting at x) and at y he should choose F (a NE for the subgame starting at y). Indeed, the only Nash equilibrium which prescribes this behavior is (O,OF). This is also the only reasonable equilibrium: in the other two Nash equilibria John is threatening to commit an action which is not in his interest if he ever ended up in that decision node. The other two equilibria contain non-credible threats.

	OO	OF	FO	FF
O	4 , 1	4 , 1	0 , 0	0 , 0
F	0 , 0	1 , 4	0 , 0	1 , 4

It appears, that subgame perfection solves the problem of non-credible threats. Soon we shall see that this is only true for perfect information games; in non-perfect information games we need further refinements of Nash equilibria (next section).

Previously, we showed by backward induction that in finitely iterated PD to defect is a unique symmetric and strict Nash equilibrium (thus an ESS). As this is a unique NE in every subgame as well, it is also a SPNE of the entire game. This is a special case of a more general result.

Theorem Consider an arbitrary iterated game $\Gamma^T(\delta)$ for $T < \infty$. Suppose that the stage game Γ has a unique pure strategy Nash equilibrium, s^*. Then $\Gamma^T(\delta)$ has a unique SPNE. In this unique SPNE $s_t = s^*$ for all $t = 1, \ldots, T$, regardless history.

Proof: We use backward induction. At T we have $s^T = s^*$. Given this we have $s^{T-1} = s^*$, and by induction $s^t = s^*$ for $t = 1, \ldots, T$. $\square$

6.4.4. *Fuzzy subgame perfection - a general theory.* We mentioned that the concept of subgame perfection is not necessarily enough for games with imperfect information. The next example, taken from Gintis' book, demonstrates this.

Example (Myron and Bob) Myron and Bob, two game-theoreticians, decide to rob ten billion dollars from a bank. They notice, however, that robbing a bank is easy, but enforcing an equal split of the money is difficult. They agree that Bob will be the gunman and will cover for Myron, who will snatch the cash, while Bob will take an incriminating photograph of the two of them robbing the bank. Unless he receives half the money, Bob will send the photograph to his lawyer, with instructions to set up an account in his name, send the account number to Myron, and send the picture to the police unless there is five billion dollars in the account within twenty four hours. If Myron and Bob will be caught stealing, they will have to give up their teaching position at the university, which is worth one billion dollars each. See Figure 7 describing the game (we draw the game-tree starting at a Bobs decision node, as the game is only interesting after Myron initially decides not to share the money).

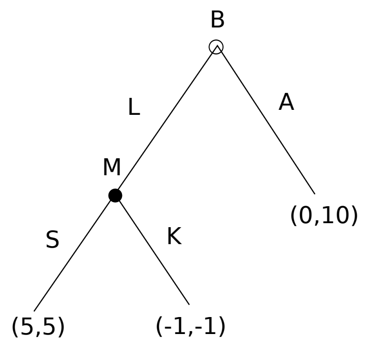


FIGURE 7. Bob can either accept (A) that Myron doesn't want to share the money, or to contact his lawyer (L). Then, Myron can either share (S) the money or keep (K) it.

Using the criterion of subgame perfection, Bob concludes that Myron must give him half the money. Satised with the arrangement, they rob the bank. Indeed, the game has two subgames, the entire game (which has (L,S) and (A,K) as Nash equilibria) and a subgame starting at Myrons decision node. In the latter subgame the Nash equilibrium is to play S, hence the SPNE of the entire game is (L,S). This is nice, it doesn't have non-credible threats and both Bob and Myron get 5 billion.

Myron, however, notices that Bob has two lawyers, Ethel and Fred, and could send the photo to either one. He therefore sends a message to Bob that he decides to keep the money even if Bob contacts his lawyers. Bob does some calculations and notices that this is indeed a subgame perfect outcome and accepts his destiny to get nothing (see Figure 8).

The game-tree describing this new game in Figure 8 shows that as there is only one subgame (the entire game), all the Nash equilibria are also SPNE. In particular,

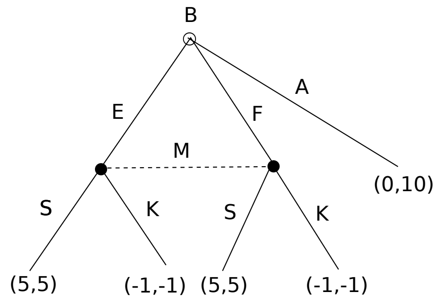


FIGURE 8. Bob can either accept (A) that Myron doesn't want to share the money, or to contact Ethel (E) or Fred (F). If Myrons information set is reached he can either share (S) the money or keep (K) it.

we have that the non-credible equilibrium (A,K), the one Myron is choosing to play, is a SPNE. (The other two pure Nash equilibria are (E,S) and (F,S), check this !)

The reason that a SPNE can be a non-credible equilibrium is that for part of the tree to be a subgame we require it to start at an information set containing only one node. This is a somewhat arbitrary requirement (well, it makes calculations bit easier), which we will drop when we define a fuzzy subgame.

Definition A *fuzzy subgame* F of a game Γ contains the following

- let μ, ν be information sets of Γ such that if a node $x \in \mu$ is a successor to a node in ν, then all $y \in \mu$ are successors of nodes in ν
- the nodes of the fuzzy subgame starting at μ consists of a new root r_ν, plus the nodes in ν and all of the successor nodes in Γ to the nodes in ν
- Nature is assigned to r_ν, with a branch from r_ν to each node in ν
- the information sets, players, and payoffs are adopted from Γ

By assigning Nature to the first information set of the fuzzy subgame, we allow the game to start at information sets that contain more than one node.

Definition We define s to be a fuzzy subgame perfect Nash equilibrium (FSPNE) if it is a Nash equilibrium in every fuzzy subgame.

The question is how do we assign probabilities to the root note r_ν? If we allow only for pure strategies it is easy. The path of play will either go through the information set ν or not. If it does, then it will go through only one node and the branch from r_ν to that node is assigned a probability 1 (consequently, all the other nodes are reached with probability 0). If the path of play doesn't go through ν we assign an arbitrary probability distribution over the whole information set. The mixed strategy case is not much more complicated but it won't be dealt in here (see Gintis).

Example (Myron and Bob) We want to know whether the non-credible SPNE (K,A) is FSPNE or not. As the strategy profile induces a path of play that doesn't reach Myrons information set, we assign a random probability distribution: with probability p Myron is at the left decision node (Bob played E) and with probability $1-p$ at the right decision node. Now, it doesn't matter how we choose p, for Myron to play K in this fuzzy subgame (i.e. the behavior prescribed by Nash equilibrium (A,K)) is not a Nash equilibrium in this fuzzy subgame: for all p it is better for Myron to play S. Hence (A,K) is not a FSPNE. The other two pure Nash equilibria (E,S) and (F,S) are.

Further refinements of Nash equilibria will be presented in the next two sections.

Lecture 10-4-2013

6.5. **Games with mistakes.** In this section we look at games where there is always a small probability to make a mistake; every action will be taken with a positive probability.

Definition The Trembling Hand Perfect Nash equilibrium (THPNE; Selten 1975) is a Nash equilibrium which is obtained as a limit point of a sequence of equilibria of disturbed games in which the mistake probabilities go to zero.

Recall, that in a NE each player's equilibrium strategy is a best response to the other player's strategies. A THPNE additionally imposes the NE should be robust to small perturbations (or mistakes) in the strategies played by other players.

Example Consider the game-tree depicted in Figure 9 and its strategic form below.

	L	R
U	1 , 1	2 , 0
D	0 , 2	2 , 2

The (pure) Nash equilibria are (U,L) and (D,R). As there is only one subgame, they are SPNE as well (what about FSPNE? exercise). Notice that for player 2 to play R doesn't make much sense, since if he ever reaches node y he is indifferent of his choice and if he ends up in x he should play L. Let us see whether the Nash equilibria are THPNE.

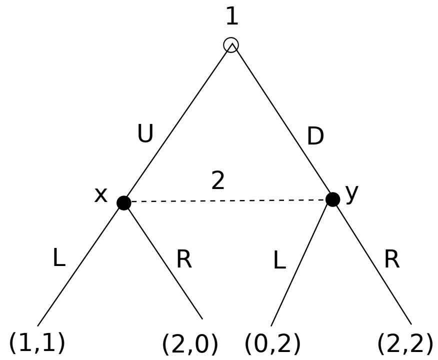


FIGURE 9.

Case **(U,L)**: What is Player 2's best reply to Player 1 if Player 1 plays D by mistake which he makes with probability ε, i.e. Player 1 plays mixed strategy $z = (1-\varepsilon, \varepsilon)$? Player 2's expected payoff by playing L is $\pi_2(z, L) = 1 + \varepsilon$ and by playing R $\pi_2(z, R) = 2\varepsilon$. For small ε playing L remains to be the best reply. What is the best reply for Player 1 if Player 2 makes mistakes with probability ξ, i.e. when Player 2 plays $y = (1-\xi, \xi)$? The expected payoffs are $\pi_1(U, y) = 1 + \xi$

and $\pi_1(U, y) = 2\xi$. For small ξ playing U thus remains to be the best reply and hence (U,L) is a THPNE.

Case **(D,R)**: Using the same procedure as above we get that the expected payoffs for Player 2 are $\pi_2(z', L) = 2 - \varepsilon$ and $\pi_2(z, R) = 2 - 2\varepsilon$, where $z = (\varepsilon, 1 - \varepsilon)$. For small mistakes the best reply is L (and not R!). Similarly we get that when Player 2 makes small mistakes the best reply for Player 1 is U (and not D). Thus (D,R) is not a THPNE.

6.5.1. *Iterated Prisoners Dilemma with mistakes.* In the 1980s, Robert Axelrod organized two computer contests for the iterated Prisoner's Dilemma (IPD) in which people could participate by submitting a strategy in the form of a computer program. Each strategy was paired with each other strategy and scored on the total points accumulated through the tournament. The very simple cooperative strategy Tit-for-Tat (see previous lecture) turned up as the winner.

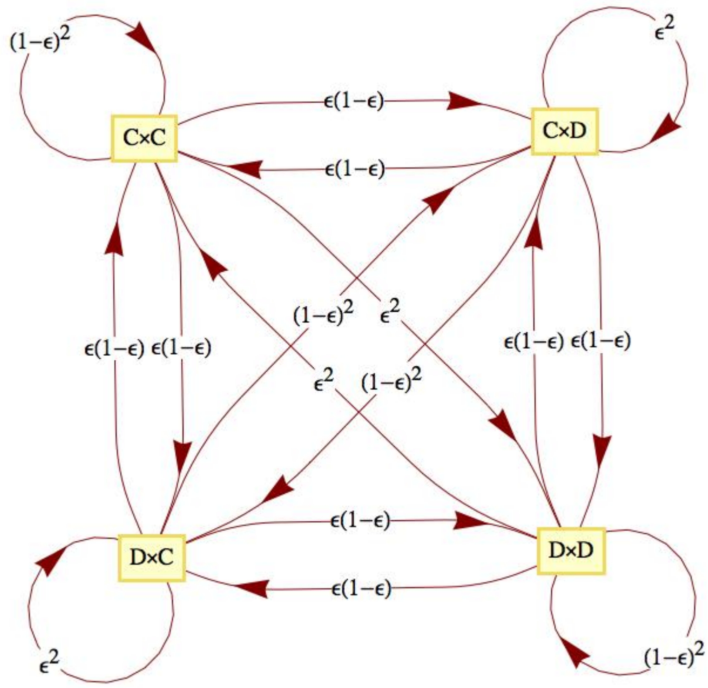
(See, e.g., the Axelrod Tournament Demonstration Software website
http://www2.econ.iastate.edu/tesfatsi/demos/axelrod/axelrodt.htm)

A weakness of TFT shows up if players can make mistakes, e.g., if they can misinterpret the opponent's move in a noisy environment. For example, in a TFT×TFT contest, if one player mistakes his opponent's 'cooperation' (C) for a 'defection' (D), the game gets trapped in a D×C→C×D→D×C etc. cycle with the corresponding low payoff to both players, at least until the next mistake occurs.

In another computer simulation with stochastic strategies in a noisy environment, Nowak and Sigmund (*Nature* (1992) **355,** 250-253) found that "Generous TFT" (GTFT) was the winner. GTFT is like TFT but "correct" mistakes by being cooperative with a given small probability after his opponent's defection. We'll now see how this works.

Depending on the current moves (C or D), and as long as we are dealing with memory-1 strategies (see previous lecture), the IPD can be in one of only four states: C×C, C×D, D×C or D×D. Contests with deterministic strategies (fixed or rule-based) in a deterministic setting therefore can never become more complicated than a four-cycle or less. In a stochastic setting, i.e., in a noisy environment which generates mistakes in the perception of the opponent's move, such limitations no longer hold.

Example. Consider TFT×TFT with a small $\varepsilon > 0$ probability of misinterpretation the opponent's move. The transition probabilities between the various states from one round to the next (conditioned on that there will be a next round) are given by the following graph:



If we put the different states of the game in the order {C×C, C×D, D×C, D×D}, then *the matrix of the graph* becomes

$$\mathbf{A}_\varepsilon = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \end{pmatrix}$$

Let further **r** be the vector of payoffs to the row-player at the different states of the game and **e** the vector of expected payoffs to the row-player calculated over all rounds for the contest if started in the respective states, i.e.,

$$\mathbf{r} = \begin{pmatrix} R \\ S \\ T \\ P \end{pmatrix} \quad \text{and} \quad \mathbf{e} = \begin{pmatrix} E_{C\times C} \\ E_{C\times D} \\ E_{D\times C} \\ E_{D\times D} \end{pmatrix}$$

Furthermore, let $\delta \in (0,1)$ be the probability that there will be a next round at all. Then, in matrix notation, we have

$$\mathbf{e} = \mathbf{r} + \delta \mathbf{A}_\varepsilon \cdot \mathbf{e}$$

which can be solved formally as

$$\mathbf{e} = (\mathbf{I} - \delta \mathbf{A}_\varepsilon)^{-1} \cdot \mathbf{r}$$

where $\mathbf{I}$ is the identity matrix. Since TFT×TFT starts with C×C, we're actually only interested in the first component of $\mathbf{e}$, which is

$$(*) \quad E_{C\times C} = \frac{1}{4}\left(\frac{P+R+S+T}{1-\delta} + \frac{P+R-S-T}{1-\delta(1-2\varepsilon)^2} - \frac{2(P-R)}{1-\delta(1-2\varepsilon)}\right)$$

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What is the proportion of time TFT×TFT spends in each of the states C×C, C×D, D×C and D×D? Let $\mathbf{p}_n$ be the probability distribution over the various states. Then, as long as the game is on, $\mathbf{p}_n$ satisfies the recurrence equation

$$\mathbf{p}_{n+1} = \mathbf{A}_\varepsilon \cdot \mathbf{p}_n$$

with initial condition $\mathbf{p}_1 = (1, 0, 0, 0)^{\mathrm{T}}$. This gives

$$\mathbf{p}_n = \mathbf{A}_\varepsilon^{n-1} \cdot \mathbf{p}_1$$

The expected number of rounds spent in the different states is

$$\mathbf{p}_1 + \delta\mathbf{p}_2 + \delta^2\mathbf{p}_3 + \cdots = \left(\sum_{k=0}^{\infty} (\delta\mathbf{A}_\varepsilon)^k\right)\mathbf{p}_1 = (\mathbf{I} - \delta\mathbf{A}_\varepsilon)^{-1} \cdot \mathbf{p}_1$$

Division by the expected number of rounds, which is $(1-\delta)^{-1}$, gives the expected proportion of time spent in each state

$$\bar{\mathbf{p}}_\varepsilon \stackrel{\text{def}}{=} (1-\delta)(\mathbf{I} - \delta\mathbf{A}_\varepsilon)^{-1} \cdot \mathbf{p}_1$$

which, if written out in full, gives

$$\bar{\mathbf{p}}_\varepsilon = \begin{pmatrix} \frac{1}{4}\left(1 + \frac{1-\delta}{1-\delta(1-2\varepsilon)^2} - \frac{2(1-\delta)}{1-\delta(1-2\varepsilon)}\right) \\ \\ \frac{\delta\varepsilon(1-\varepsilon)}{1-\delta(1-2\varepsilon)^2} \\ \\ \frac{\delta\varepsilon(1-\varepsilon)}{1-\delta(1-2\varepsilon)^2} \\ \\ \frac{\delta\varepsilon^2(1-\delta(1-2\varepsilon))}{(1-\delta(1-2\varepsilon)^2)(1-\delta(1-2\varepsilon))} \end{pmatrix}$$

Expansion of $\bar{\mathbf{p}}_\varepsilon$ into terms of different orders of ε gives an idea of the distribution if mistakes are rare:

$$\bar{\mathbf{p}}_\varepsilon = \begin{pmatrix} 1 - \frac{2\delta\varepsilon}{1-\delta} + \mathrm{O}(\varepsilon^2) \\ \\ \frac{\delta\varepsilon}{1-\delta} + \mathrm{O}(\varepsilon^2) \\ \\ \frac{\delta\varepsilon}{1-\delta} + \mathrm{O}(\varepsilon^2) \\ \\ \mathrm{O}(\varepsilon^2) \end{pmatrix}$$

Thus, for small ε, TFT×TFT spends most of its time in mutual cooperation and only a tiny fraction of its time in mutual defection.

Example. Consider GTFT×TFT. The transitions between the different states of the game are now double stochastic: the players may or may not make a mistake (with probabilities ε and $1-\varepsilon$) and, independently, GTFT may or may not forgive his opponent's defection (with probabilities γ and $1-\gamma$).

For example, the probability of moving from the state C×D (where GTFT cooperates and TFT defects) to the state C×C is equal to: $((1-\varepsilon)\gamma+\varepsilon) \times (1-\varepsilon)$, i.e., the probability that GTFT (*i*) does not make a mistake in the perception of his opponent's move but nonetheless forgives him or (*ii*) does make a mistake in perception by thinking his opponent was actually cooperating (both possibilities lead to GTFT to continue to cooperate also in the next round), multiplied by the probability that TFT does not make a mistake and thus switches to cooperation in the next round.

The matrix of the transition probabilities is a bit more complicated than previously and requires some concentration to compute (exercise).

7. Signalling games

We have seen that asymmetries due to different roles recognized by both players can often resolve a situation of conflict without resorting to an escalated fight. But how does a player know the role of its opponent? We have seen that if the process of assessing the opponent's role (e.g., its strength) is too costly, it may not be worth to find out. What if the players just signal to their opponent what role they are in, e.g., by giving a demonstration of strength. But is this demonstration reliable? It might pay the sender to exaggerate its claim. Does it pay the receiver of the signal to believe the sender of the signal? If not, then there is no point in signaling to begin with. This is the conundrum that signaling games address.

It turns out that there are plenty of signals in nature.
Examples (a) Spiders *Agelenopsis aperta* may compete for location to put up a spinning web (Riechter 1978, Gintis 2000). They use various signals to reveal their strength/size (e.g. vibrating the web, standing on two legs) in order to avoid unnecessary and costly fight. These signals seem to be reliable signals, as the weaker/smaller opponent usually retreats. (b) sexual signals: As females are usually the ones to choose a mating partner, they are under selection pressure to choose the best male: they would like to distinguish between good and bad males, i.e. they would like to find some reliable signals of male quality. On the other hand, males would like to signal their quality (either reliably or not!) to increase their chances to be chosen for mating. This type of conflict has sometimes led to rather complex and elaborate signals: Doucet and Montgomerie 2003 showed that in the case of male satin bowerbirds (*Ptilonarhynchus violaceus*) the bower

("love nest") quality (coloration, symmetry) predicts ectoparasite load and body size, whereas ultraviolet plumage coloration predicts the intensity of infection from blood parasites, feather growth rate, and body size. Females should then check the quality of bowers and the coloration of the plumage of the male before making her choice, as they are honest signals of males quality! The reader is encouraged to check on youtube how extraordinary these bowers are ☺

In short, a signaling game contains the following steps

- Player 1 has private information (i.e. what type he is, strong, weak etc.).
- Player 1 can take an action based on the private information. This action will be called a signal and therefore Player 1 will be called the *sender* of the signal.
- Player 2 observes the Player 1's action (signal). Player 2 will be called the *receiver* of the signal.
- Player 2 takes his/her own action to respond to the signal.

There will be potentially three types of solutions: (i) *pooling*, where all the types of Senders play the same signal. In this case the signal doesn't reveal the private information of the Sender (ii) *separating*, different types of Senders choose different signals. In this case the signal reveals the type of the Sender. (iii) *semi-separating*, where one type plays a pure strategy and other plays a mixed strategy.

7.0.2. *A more formal description of signaling games.* The set of all possible types is denoted by $\mathcal{T}$. As mentioned earlier, the type of the sender is known to itself, but not to receiver. The receiver only knows the probability distribution of types in the population as a whole, which is denoted by $P(t)$, i.e., the probability that a randomly selected sender is of type t. In Bayesian terms the distribution P is called the "prior distribution".

The set of all possible signals is denoted by $\mathcal{S}$. Signals may indicate senders type truthfully or not. The signal may, e.g., be a simple statement like "I am very strong" or a demonstration of strength or whatever.

Depending on the signal, the receiver chooses an action such as, e.g., fight, flee, etc. The set of all possible actions is denoted by $\mathcal{A}$. For now, we shall assume that the sets $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{A}$ are finite.

The payoffs to the sender and the receiver are given functions $\pi_{\mathrm{S}}\colon \mathcal{T}\times\mathcal{S}\times\mathcal{A}\to\mathbb{R}$ and $\pi_{\mathrm{R}}\colon \mathcal{T}\times\mathcal{S}\times\mathcal{A}\to\mathbb{R}$.

Strategy of the sender: A pure strategy of the sender is a function $\sigma\colon \mathcal{T}\to\mathcal{S}$ that assigns to every type $t\in\mathcal{T}$ a signal $s\in\mathcal{S}$ such that $s=\sigma(t)$ is the signal given by a sender of type t. A mixed strategy of the sender is a conditional probability distribution over $\mathcal{S}$ given the type of the sender and is denoted by $P_{\mathrm{S}}(s|t)$, i.e., the probability of giving the signal s if the sender's type is t.

Strategy of the receiver: A pure strategy of the receiver is a function $\alpha\colon \mathcal{S} \to \mathcal{A}$ that assigns to every signal $s \in \mathcal{S}$ an action $a \in \mathcal{A}$ such that $a = \alpha(s)$ is the action of the receiver in response to the signal s. A mixed strategy of the receiver is a conditional probability distribution over $\mathcal{A}$ given the the signal received and is denoted by $P_{\mathrm{R}}(a|s)$, i.e., the probability of choosing action a of the signal is s.

The expected payoff E_{S} to a sender of type t with strategy P_{S} against an receiver with strategy P_{R} is

$$E_{\mathrm{S}}(P_{\mathrm{S}}, P_{\mathrm{R}}|t) = \sum_{s\in\mathcal{S}} \sum_{a\in\mathcal{A}} \pi_{\mathrm{S}}(t, s, a) P_{\mathrm{S}}(s|t) P_{\mathrm{R}}(a|s) \tag{18}$$

The expected payoff E_{S} to a receiver of the signal s with strategy P_{R} against an receiver with strategy P_{S} is

$$E_{\mathrm{R}}(P_{\mathrm{S}}, P_{\mathrm{R}}|s) = \sum_{t\in\mathcal{T}} \sum_{a\in\mathcal{A}} \pi_{\mathrm{R}}(t, s, a) P(t|s) P_{\mathrm{R}}(a|s), \tag{19}$$

where $P(t|s)$is the probability that the sender is of type t given that he signals s. It will be called a *belief* of the receiver and to make the notation more distinct from the mixed strategy notation we write $\mu(t|s) \equiv P(t|s)$.

Lecture 17-4-2013

Definition A player is called *sequentially rational* at a history if he plays a best reply to a belief conditional on being at that history. Formally, we say that an assessment $(\hat{P}_s, \hat{P}_R; \hat{\mu})$ is sequentially rational if

$$\begin{cases} E_{\rm S}(P_{\rm S}, \hat{P}_{\rm R}|t) \leq E_{\rm S}(\hat{P}_{\rm S}, \hat{P}_{\rm R}|t) & \forall t \quad \forall P_{\rm S} \\ \\ E_{\rm R}(\hat{P}_{\rm S}, P_{\rm R}|s) \leq E_{\rm S}(\hat{P}_{\rm S}, \hat{P}_{\rm R}|s) & \forall s \quad \forall P_{\rm R} \end{cases} \tag{20}$$

Definition An assessment $(\hat{P}_s, \hat{P}_R; \hat{\mu})$ is a *Perfect Bayesian Nash Equilibrium* if it is sequentially rational and μ is computed from $(\hat{P}_s, \hat{P}_R)$ by Bayes' rule in any information set if possible, i.e.

$$\mu(t|s) = \frac{P_{\rm S}(s|t)P(t)}{\sum_{t'\in\mathcal{T}} P_{\rm S}(s|t')P(t')} \tag{21}$$

for $\sum_{t'\in\mathcal{T}} P_{\rm S}(s|t')P(t') > 0$ and if $\sum_{t'\in\mathcal{T}} P_{\rm S}(s|t')P(t') = 0$, then $\mu(t|s)$ is any probability distribution.

Remarks: A belief $\mu(t|s) = P(t|s)$ can be seen as an updated probability from $P(t)$. $P(t)$ gives the probability that sender is of type t before the signal is observed, and $\mu(t|s)$ gives the probability (a belief) that sender is of type t after the signal s is observed. It hence gives the probability that he is at a particular decision node, or more appropriately, his beliefs that he is at a particular node.

We will consider games with two types of senders $t_1, t_2 \in \mathcal{T}$, two signals $s_1, s_2 \in \mathcal{S}$ and two actions $a_1, a_2 \in \mathcal{A}$. Game-trees then may be drawn as in Figure 10.

Consistent with the definitions above, we can classify the possible solutions as follows

- *Pooling equilibrium:* signals are the same for different types, i.e. $\sigma(t_1) = \sigma(t_2)$
- *Separating equilibrium:* signals are different for different types, i.e. $\sigma(t_1) \neq \sigma(t_2)$
- *Semi-separating equilibrium:* if it is neither of the above.

The next example will hopefully demonstrate how to find possible PBNE. It was first presented in the seminal paper of Cho and Kreps (1987), motivated by the book "Real Men Don't Eat Quiche". The derivation of the first solution will be very lengthy, so it will require some time for the reader to chew through it. Despite of its length, hopefully it will serve as a guide how to find solution to games of

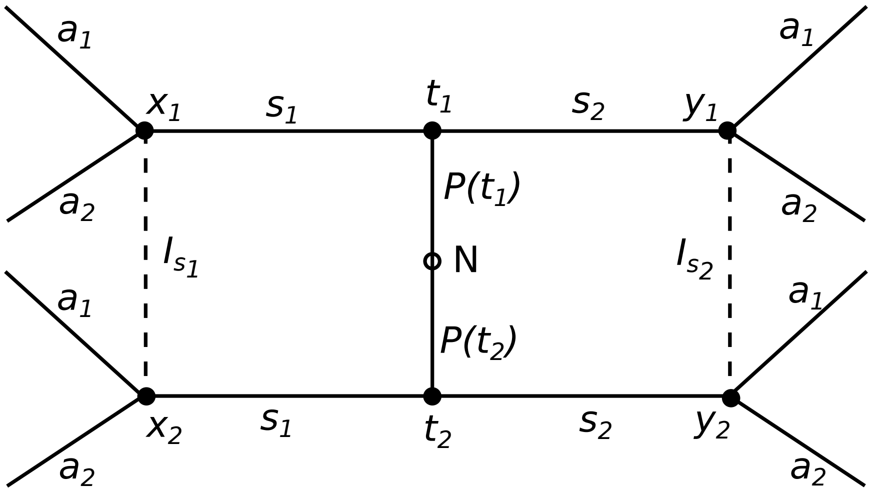

FIGURE 10. Game-tree for signalling games. In addition, at each terminal node we write the payoffs for the sender and the receiver; for example, after moves t_1, s_1, a_1 are made the payoffs are simply $(\pi_S(t_1, s_1, a_1), \pi_R(t_1, s_1, a_1))$ (the upper left corner). Given the signal s_1 the beliefs at nodes x_1 and x_2 are $\mu(t_1|s_1)$ and $\mu(t_2|s_1)$, respectively. That is, if signal s_1 is observed by the receiver (i.e the receiver is at the information set I_{s_1}), he/she beliefs is at node x_1 with probability $\mu(t_1|s_1)$ and at node x_2 with probability $\mu(t_2|s_1)$.

this type.

Example (Beer & Quiche): Leaders of two great nations are having final negotiation to decide whether they go to war with each other or not. Leader 1 doesn't want to go to war under any circumstances. Leader 2 wants to go to war if the Leader 1 is weak, but he doesn't want to go to war if the Leader 1 is strong. Now, Leaders have breakfast together before the negotiations, and on the menu there is Beer and Quiche (a pie). It is in the interest of Leader 1 to choose his breakfast such, that Leader 2 would think the Leader 1 is strong, and hence there would be no war. What should Leader one choose for breakfast?

The game-tree describing this situation is depicted in Figure 11. Leader 1, the sender, can be of two types t_w (weak) or t_k (strong, k stands for kova jätkä). With probability 0.1 the sender is weak and probability 0.9 he is strong. In either case, he can choose for breakfast beer (b) or quiche (q). These are the signals he sends to the receiver. Receiver observes the signal and either goes to war (f) or not (n).

We will look only for pure PBNE strategies for the sender, i.e. we are only interested in PBNE which contain either a strategy **(a)** $(\sigma(t_w), \sigma(t_k)) = (q, q)$ **(b)**

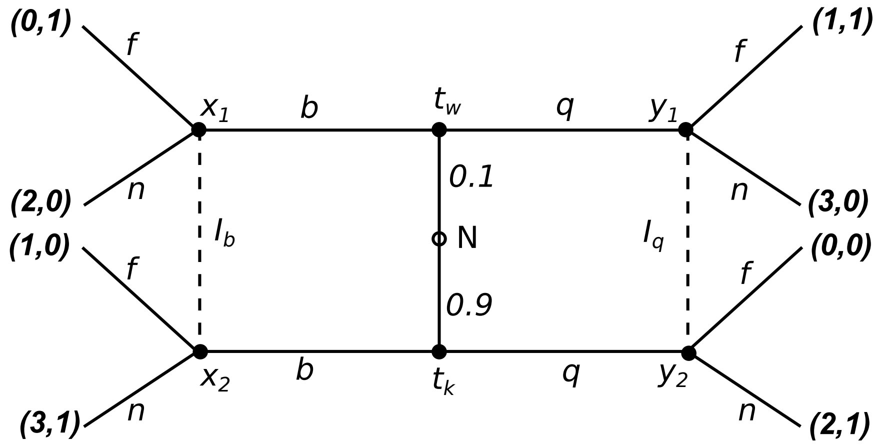

FIGURE 11.

$(\sigma(t_w), \sigma(t_k)) = (b, b)$ **(c)** $(\sigma(t_w), \sigma(t_k)) = (q, b)$ or **(d)** $(\sigma(t_w), \sigma(t_k)) = (b, q)$. The first two are the pooling equilibria and the second two are separating.

(a) Can $(\sigma(t_w), \sigma(t_k)) = (q, q)$ be part of a PBNE? In other words, can we find strategies and beliefs for the receiver such that they are the best replies to (q, q) and that (q, q) is the best reply to receivers strategies and beliefs? Let us calculate the beliefs at information sets I_q and I_b.
First consider I_q: If (q, q) is part of a PBNE, we reach the information set I_q with probability 1 because the sender has quiche for breakfast independently of his type. If (q, q) is part of a PBNE, receiver knows this, and hence he beliefs that given signal q (which is sent with probability 1) he is at node y_1 with probability 0.1 and at node y_2 with probability 0.9, since these are the probabilities that the sender is of type t_w and t_k, resp.. More formally, we have $(P(t_w), P(t_k)) = (0.1, 0.9)$ and $(P_S(b|t_w), P_S(q|t_w)) = (0, 1)$, $(P_S(b|t_k), P_S(q|t_k)) = (0, 1)$ and from (21) we get $\mu(t_w|q) = 0.1$ and $\mu(t_k|q) = 0.9$.
What is the best reply for the receiver given these beliefs and the senders strategy (q, q)? The best reply is n, because the expected payoff when playing n is $\pi_R(t_w, q, n)\mu(t_w|q) + \pi_R(t_k, q, n)\mu(t_k|q) = 0.9$ and when playing f is $\pi_R(t_w, q, f)\mu(t_w|q) + \pi_R(t_k, q, f)\mu(t_k|q) = 0.1$. Alright, we found that the receiver should play n after observing signal q.
Now consider I_b: As $P_S(b|t_w) = 0 = P_S(b, |t_k)$, we can't apply Bayes' rule at this information set. We are then free, if we can, to choose $\mu(\cdot|b)$ such, that it will support (q, q) to be part of PBNE. That is, we would like to choose the beliefs (and receivers strategies!) such, that the sender would not want to deviate from (q, q). *If* sender is of type t_k, he signals q and since receivers best reply was n the payoff to the sender is $\pi_S(t_k, q, n) = 2$. If receiver would play n also at the information set I_b, then sender of type t_k would want to play b instead of q because

$\pi_S(t_k, b, n) = 3$. We then want that the receiver would play f at the information set I_b, because $\pi_S(t_k, b, f) = 1$. Note that we are fully aware that the information set I_b is never reached if the senders strategy (q, q) is played, but PBNE requires that we define NE actions even at off-equilibrium paths! Our task is then to find beliefs for the receiver at the information set I_b such that the receiver will gain the highest payoff by playing f, and not n.
The receiver will play f at I_b iff his expected payoff is greater by playing f than by playing n, that is, if $\pi_R(t_w, b, f)\mu(t_w|b) + \pi_R(t_k, b, f)\mu(t_k|b) = 1 - \mu(t_k|b) \geq \pi_R(t_w, b, n)\mu(t_w|b) + \pi_R(t_k, b, n)\mu(t_k|b) = \mu(t_k|b)$. We have then that if $\mu(t_k|b) \leq \frac{1}{2}$, the best reply for the receiver at information set I_b is f, and consequently, the sender of type t_k doesn't want to deviate from the strategy q. But what if the sender is of type t_w? Given the belief we just calculated and the best reply f of the receiver at the information set I_b, would the sender of type t_w want to deviate and play b instead? Given the above beliefs we have that the receiver plays n at I_q and f at I_b, and because $\pi_S(t_w, q, n) = 3 \geq \pi_S(t_w, b, f) = 0$, the best response for the sender of type t_w is q. Finally, solution is ready.
The assessment

$$\Big[(\sigma(t_w), \sigma(t_k)), (\alpha(b), \alpha(q)); \big((\mu(t_w|b), \mu(t_k|b), (\mu(t_w|q), \mu(t_k|q))\big)\Big] \tag{22}$$

$$= [(q, q), (f, n); ((\gamma, 1-\gamma), (0.1, 0.9))], \quad \text{for } \gamma > \frac{1}{2} \tag{23}$$

is a PBNE.
What is the interpretation of this PBNE? If Leader 2 beliefs that Leader 1 is more likely to be weak *if* he would happen to have beer for breakfast, then (q, q) is part of a PBNE. Since Leader 1 *will* have quiche for breakfast Leader 2 will not go to war.

The case (b) we leave as an exercise.

(c) Can a separating equilibrium $(\sigma(t_w), \sigma(t_k)) = (q, b)$ be part of a PBNE? We have $(P_S(b|t_w), P_S(q|t_w)) = (0, 1)$, and $(P_S(b|t_k), P_S(q|t_k)) = (1, 0)$ and from Bayes' rule we get $(\mu(t_w|b), \mu(t_k|b)) = (0, 1)$ and $(\mu(t_w|q), \mu(t_k|q)) = (1, 0)$. Indeed, if (q, b) is part of a PBNE, receiver knows this (not necessarily of course, but he can hypothesize as we are doing!) and will know precisely at which node he is after observing a signal: if he observes beer, he knows the sender is t_k and if he observes quiche he knows the sender is t_w. The best reply to b is then n, because $\pi_R(t_k, b, n) = 1 > \pi_R(t_k, b, f) = 0$ and the best reply to q is f because $\pi_R(t_w, q, f) = 1 > \pi_R(t_w, q, n) = 0$.
Since observing beer makes the receiver play n, the sender of type t_w ought to change from q to play b because $\pi_S(t_w, b, n) = 2 > \pi_S(t_w, q, f)$. Hence (q, b) is *not* a best reply to the beliefs and the strategy of the receiver, and hence is not part of a PBNE.

The case (d) we leave as an exercise.

Lecture 22-4-2013

8. Multi-stage games

A general *multi-stage game* Γ (also called a stochastic game or a Markov game) is a finite or infinite sequence of rounds of the same or different stage-games $\Gamma_1, \ldots, \Gamma_N$. In each round the payoff is a real payoff plus a "ticket" to the same or another stage-game of the game played against the same or another opponent. The action sets $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ of the row- and the column-player for different stages may be different or the same. The payoff functions $\pi^i = (\pi^i_1, \pi^i_2) : X_i \times Y_i \to \mathbb{R}^2$, too, have to be specified for each stage-game of the game separately. We thus have

$$\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_N)$$

with strategy sets

$$X = X_1 \times X_2 \times \cdots \times X_N \quad \text{and} \quad Y = Y_1 \times Y_2 \times \cdots \times Y_N$$

and overall payoff function

$$\pi : X \times Y \to \mathbb{R}^2$$

and a solution concept, which we shall take to be the ESS. The rub is how to calculate the overall payoff function π from the payoff functions π^i of the separate stages.

Remarks: (i) Strategies in multi-stage games are defined as in ietrated games with the obvious extension: the action sets might differ from stage-game to stage-game (and hence also from round to round). Here, we will only consider strategies where the action is same when the same stage-game is played.
(ii) An *iterated game* is a special form of a multi-stage game $\Gamma = (\Gamma_1)$. For example, in the iterated Prisoner's Dilemma we have $X_1 = Y_1 = \{C, D\}$, and the payoff functions π^1 are defined by the payoff matrix

Γ_1	C	D
C	$R + \delta\,\Gamma_1,\ R + \delta\,\Gamma_1$	$S + \delta\,\Gamma_{i+1},\ T + \delta\,\Gamma_1$
D	$T + \delta\,\Gamma_1,\ S + \delta\,\Gamma_1$	$P + \delta\,\Gamma_1,\ P + \delta\,\Gamma_1$

where $\delta \in (0, 1)$ is the discounting factor (i.e., the probability of that there is a next round) and Γ_1 represents the "ticket" to the next round of the game.

Example 1 Consider the Hawk-Dove game where the looser of a H×H contest must skip one round, or more, to recover (R) from its injuries. The winner, on the other hand, continues with another round of the Hawk-Dove game against another opponent. The strategy of the new opponent is the same as that of the previous one, because we want to calculate the payoff to a rare invader strategy against the resident strategy. This situation can be modeled as a two-stage game $\Gamma = (\Gamma_1, \Gamma_2)$ where Γ_1 is the Hawk-Dove game with payoff matrix

Γ_1	H	D
H	$\frac{1}{2}(R+\delta\Gamma_1)+\frac{1}{2}\delta\Gamma_2$	$R+\delta\Gamma_1$
D	$\delta\Gamma_1$	$\frac{1}{2}R+\delta\Gamma_1$

(symmetric game: payoffs to the row player)

and Γ_2 is a recovery round with payoff matrix

Γ_2	
R	$\varepsilon\delta\Gamma_1+(1-\varepsilon)\delta\Gamma_2$

(one-player game)

where ε is the probability of recovery and δ is the probability of playing another round. Notice that Γ_2 has only one player: the loser of the H×H contest. Also notice that the cost of injury (previously denoted by C) in Γ_1 is now replaced by not being able to play and gather resources for one or more rounds.

If only pure strategies are allowed, then strategy sets of the overall game $\Gamma = (\Gamma_1, \Gamma_2)$ are $X = Y = \{$(H,R),(D,R)$\}$. The calculation of the overall payoffs is not much different from how we calculated payoffs for iterated games.

(H,R)×(H,R) – Let E_1 and E_2 denote the payoff to the row-player if starting with, respectively, Γ_1 or Γ_2. Then

$$\begin{cases} E_1 = \frac{1}{2}(R+\delta E_1)+\frac{1}{2}\delta E_2 \\ \\ E_2 = \varepsilon\delta E_1 + (1-\varepsilon)\delta E_2 \end{cases}$$

from which we solve

$$E_1 = \frac{R\big(1-\delta(1-\varepsilon)\big)}{(1-\delta)\big(2-\delta(1-2\varepsilon)\big)}$$

Calculation of the payoffs for the other strategy combinations is trivial (see lecture on iterated games) because they do not involve switching between the different stages of the game. The overall payoff matrix then becomes

Γ	(H,R)	(D,R)
(H,R)	$\frac{R(1-\delta(1-\varepsilon))}{(1-\delta)(2-\delta(1-2\varepsilon))}$	$\frac{R}{1-\delta}$
(D,R)	0	$\frac{R}{2(1-\delta)}$

(symmetric game: payoffs to the row player)

It follows that (H,R) is evolutionarily stable. Since the support of an ESS cannot be a subset of the support of another ESS, (H,R) is the only ESS of the game, and so we do not have to look for a mixed ESS.

Example 2 As a variation on the previous game, assume that the winner of a Hawk×Hawk contest is not paired with another opponent, but simply takes (T) the resource each round until his opponent has recovered (R) and can play again. We then have a two-stage game $\Gamma = (\Gamma_1, \Gamma_2)$ where Γ_1 is a symmetric Hawk-Dove game:

Γ_1	H	D
H	$\frac{1}{2}(R + \delta\Gamma_2^{\text{col}}) + \frac{1}{2}\delta\Gamma_2^{\text{row}}$	$R + \delta\Gamma_1$
D	$\delta\Gamma_1$	$\frac{1}{2}R + \delta\Gamma_1$

(symmetric game: payoffs to the row player)

Notice that the winner of the H×H contest in Γ_1 gets a "ticket" to play the column player in Γ_2 while the looser gets a "ticket" to play the row player. So, Γ_2 is an asymmetric game with two roles: injured (row player) and uninjured (column player):

Γ_2	T
R	$\varepsilon\delta\Gamma_1 + (1-\varepsilon)\delta\Gamma_2^{\text{row}}\ ,\ R + \varepsilon\delta\Gamma_1 + (1-\varepsilon)\delta\Gamma_2^{\text{col}}$

(asymmetric game: injured (row player), not injured (column player))

The strategy sets are

$$X = Y = \left\{ \left(\text{H}, \begin{pmatrix} \text{T} \\ \text{R} \end{pmatrix} \right), \left(\text{D}, \begin{pmatrix} \text{T} \\ \text{R} \end{pmatrix} \right) \right\}$$

Here is how we calculate the payoffs for the overall game Γ:

(H,(T,R))×(H,(T,R)) – Let E_1 denote the payoff to the row-player if starting with Γ_1, and let E_2^{row} and E_2^{col} be the payoff to players if starting with Γ_2 in the role of row player and column player, respectively. Then

$$\begin{cases} E_1 &= \frac{1}{2}(R + \frac{1}{2}\delta E_2^{\text{col}}) + \frac{1}{2}\delta E_2^{\text{row}} \\ E_2^{\text{row}} &= \varepsilon\delta E_1 + (1-\varepsilon)\delta E_2^{\text{row}} \\ E_2^{\text{col}} &= R + \varepsilon\delta E_1 + (1-\varepsilon)\delta E_2^{\text{col}} \end{cases}$$

If we write $E_2 \overset{def}{=} \frac{1}{2}(E_2^{\text{row}} + E_2^{\text{col}})$, then the above simplifies to

$$\begin{cases} E_1 &= \frac{1}{2}R + \delta E_2 \\ E_2 &= \frac{1}{2}R + \varepsilon\delta E_1 + (1-\varepsilon)\delta E_2 \end{cases}$$

from which we solve

$$E_1 = \frac{R}{2(1-\delta)}$$

The payoffs for the other strategy combinations stays the same as in the previous example. The overall payoff matrix thus becomes

Γ	(H,(T,R))	(D,(T,R))
(H,(T,R))	$\frac{R}{2(1-\delta)}$	$\frac{R}{1-\delta}$
(D,(T,R))	0	$\frac{R}{2(1-\delta)}$

(symmetric game: payoffs to the row player)

It follows that (H,(T,R)) is evolutionarily stable. Since the support of an ESS cannot be a subset of the support of another ESS, this is also the only ESS of the game.

9. Replicator dynamics

Evolutionary stability is about immunity of a resident population of a given strategy against invasion by an initially rare mutant strategy. Invasion is essentially a population dynamical concept. So far we have been rather implicit about the connection between games and the underlying population dynamics.

In the setup for the ESS criterion, individuals were assumed to use pure or mixed strategies. For example, each individual plays with probability p_i a pure strategy x_i. In here, we interpret a mixed strategy as a *population state*, each component p_i representing the fraction of individuals who use only the pure strategy x_i. Notice, that if the population is large, well mixed and the opponent is chosen randomly, in both interpretations the probability that your opponent uses strategy x_i is p_i.

Let us suppose there exists a mechanism under which the fraction of individuals which play a certain pure strategy can change continuously. For example, if the mechanism is reproduction, then due to natural selection the individuals who carry good genes will have on average more offspring than others and therefore in the next generation the number of individuals who carry those genes will relatively increase. In this case the payoff is interpreted as fitness (probability of survival times reproduction times fecundity) and strategies as genes (alleles). Other mechanism how strategies spread may be imitation (see Sigmund 2010).

The standard way to model the above is to use the so-called *replicator equation* (Taylor and Jonker 1978): let $x_1, \ldots, x_k \in X$ be strategies, and let $p_1, \ldots, p_k \geq 0$ with $\sum_i p_i = 1$ be the corresponding relative frequencies in a given population.

Then the expected payoff to a player with strategy x_i against a randomly selected opponent is

$$w_i \stackrel{\text{def}}{=} \sum_{j=1}^{k} \pi_1(x_i, x_j) p_j$$

and the expected payoff to a randomly selected player against a randomly selected opponent is

$$\bar{w} \stackrel{\text{def}}{=} \sum_{i=1}^{k} \sum_{j=1}^{k} \pi_1(x_i, x_j) p_i p_j$$

The replicator equation is the equation

$$\frac{dp_i}{dt} = p_i\big(w_i - \bar{w}\big) \quad (i = 1, \ldots, k)$$

describing the continuous change of the relative frequencies p_i.

Using the replicator equation we can easily derive the ESS condition given previously (not done here). The key issue is stability. Consider two pure strategies x, y with frequencies $p, 1-p$, respectively. Strategy x is said to be an ESS, if strategy y when rare can't increase in frequency. That is, the equilibrium point $p = 1$ must be in some sense stable as p can't decrease in the neighborhood of $p = 1$ (i.e. rare mutants can't increase in frequency). For example an asymptotic stability of an equilibrium point guarantees that frequencies will approach the original equilibrium value after a small perturbation.
We have the following results (i) Every ESS is asymptotical stable (Weibull 1995) (ii) If Nash equilibrium is *not* a Trembling hand perfect Nash equilibrium then it is not asymptotically stable (Bomze 1986). Without any further discussion, these results demonstrate that stability of a solution is the fundamental difference between an ESS and a NE concepts. See for example Weibull 1995 for an excellent account on the relationship between various stability and solution concepts.

Lecture 24-4-2013

In the last lecture we give a summary of the course and organize an IPD tournament (instructions are given on the website).