Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 8
19-23.03.2012

1. Consider the compact group S^1 (with the standard multiplication of complex numbers). Let $p\colon \mathbb{R} \to S^1$ be defined by

$$p(t) = e^{2\pi it} = \cos(2\pi(t)) + i\sin(2\pi(t)).$$

Prove that the Haar integral for S^1 is given by the formula

$$\int_{S^1} f(x)dx = (R)\int_0^1 (f \circ p)(t)dt$$

for all continuous $f\colon S^1 \to \mathbb{R}$. Here the integral of the right side is the usual Riemann integral on reals.

2. Let $n \in \mathbb{N}$ and define $f\colon S^1 \to S^1$ by $f(z) = z^n$. Use the previous exercise to calculate the value of the Haar integral $\int f$ (where we think of f as mapping $f\colon S^1 \to \mathbb{R}^2$).

3. Suppose V is an n-dimensional vector space, $n \in \mathbb{N}$. Let $A\colon V \to \mathbb{R}^n$ be a linear isomorphism. Define **Euclidean topology** on V by requiring that A is a homeomorphism.
a) Show that this topology is uniquely defined and does not depend on the choice of the isomorphism A.
b) Suppose V, W are finite-dimensional vector spaces and $L\colon V \to W$ is a linear mapping. Show that L is continuous with respect to the Euclidean topologies on V and W.
(Hint: prove b) first and apply it to the identity mapping to derive a)).
c) Let $GL(V)$ be the group of all linear isomorphisms $L\colon V \to V$ (with respect to the composition of mappings) and let $v = \{v_1, \ldots, v_n\}$ be an (ordered) basis of V. Show that the mapping

$$\mu_v\colon GL(V) \to GL(n, \mathbb{R})$$

defined by $\mu_v(A) = [A]_{v,v}$ (the matrix of A with respect to the basis v) is an isomorphism of groups and the topology co-induced by μ_v in the group $GL(V)$ does not depend on the choice of the basis of V. Show that $GL(V)$ is a topological group with respect to this topology (also referred to as the Euclidean topology of the group $GL(V)$).

4. Suppose G is a group and V is a finite-dimensional vector space. No topologies are considered (yet). A homomorphism of groups $\phi\colon G \to GL(V)$ is called **a linear representation** of G in V. A mapping $\Phi\colon G \times V \to V, \Phi(g, v) = gv$ is called **linear action** of G on V if it satisfies algebraic conditions of the action i.e.

i)$ev = v$ for all $v \in V$, e neutral element of G,
ii)$(gg')v = g(g'v)$ for $g, g' \in G, v \in V$
and also
iii)the mapping $\Phi_g\colon V \to V$ defined by $v \mapsto gv$ is linear for every $g \in G$.

a) Suppose $\phi\colon G \to GL(V)$ is a linear representation of G in V. Define $\hat{\phi}\colon G \times V \to V$ by $\hat{\phi}(g, v) = \phi(g)(v)$. Prove that the correspondence $\phi \to \hat{\phi}$ is a bijection from the set of all linear representations of G and the set of all linear actions of G on V.

b) Now suppose G is a topological group and equip V with its Euclidean topology. Prove that a linear representation ϕ of G in V is continuous (with respect to the Euclidean topology in $GL(V)$) if and only if the corresponding linear action $\hat{\phi}$ is continuous.

c) Suppose G is a topological group and V is a finite-dimensional vector space with the Euclidean topology. Suppose $\Phi\colon G \times V \to V, \Phi(g, v) = gv$ is a linear action of G on V as defined above (not assumed continuous). Deduce that the following conditions are equivalent:
i) Φ is continuous.
ii) For every fixed $v \in V$ a mapping $\Phi_v\colon G \to V$ defined by $\Phi_v(g) = gv$ is continuous.
iii) There exists a basis $\{v_1, \ldots, v_n\}$ of V such that Φ_{v_i} is continuous for all $i = 1, \ldots, n$.

5. Suppose G is a topological group, V is a finite-dimensional vector space and $\phi\colon G \to V$ is a continuous linear representation. A mapping $f\colon Map(G, \mathbb{R}) \to \mathbb{R}$ is called **matrix coefficient** of the representation ϕ is there exists $v \in V$ and linear $L\colon V \to \mathbb{R}$ such that

$$f(g) = L(\phi(g)(v))$$

for all $g \in G$. The vector subspace of $Map(G, \mathbb{R})$ spanned by all matrix coefficients of ϕ will be denoted $\mathcal{M}_\phi$.
a) Choose a basis in $v_1, \ldots, v_n$ in V and represent all linear mapping $\phi(g)$ in that basis as matrices:

$$\phi(g) = \begin{bmatrix} \phi(g)_{11} & \ldots \phi(g)_{1n} \\ \ldots & \ldots \\ \phi(g)_{n1} & \ldots \phi(g)_{nn} \end{bmatrix}$$

Prove that every mapping $\phi(g)_{ij}$ obtained from the corresponding coefficient of such matrix is a matrix coefficient of ϕ (which explains terminology).

b) Prove that every matrix representation of ϕ can be written as a linear combination of matrix coefficients $\phi(g)_{ij}$ defined as above. Conclude that $\mathcal{M}_\phi$ is finite-dimensional and in fact $\dim \mathcal{M}_\phi \leq (\dim V)^2$.

6. Suppose f is a matrix coefficient of a continuous linear representation $\phi\colon G \to V$ and suppose $g \in G$. Show that $R_g f$ and $L_g f$ are also matrix coefficients of

ϕ. Conclude that the vector subspace of $Map(G, \mathbb{R})$ spanned by the set

$$\{R_g f \mid g \in G\}$$

is finite-dimensional and the same is true for the vector subspace spanned by the set

$$\{L_g f \mid g \in G\}.$$

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.