Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 7
12-16.03.2012

1. Suppose G is a topological group, $n \in \mathbb{N}$. Let
$$V = \{f\colon G \to \mathbb{R}^n \mid f \text{ is continuous}\}.$$
Then V is a vector space with addition and scalar multiplication defined pointwise,
$$(f + f')(x) = f(x) + f'(x),$$
$$(af)(x) = af(x).$$
Let $h \in G$ be fixed. Define $R_h, L_h\colon V \to V$ by
$$R_h(f)(g) = f(gh),$$
$$L_h(f)(g) = f(hg).$$
Prove that R_h and L_h are linear isomorphisms of V and we have the following identities
$$R_h \circ R_{h'} = R_{hh'},$$
$$L_h \circ L_{h'} = L_{h'h},$$
$$R_e = \mathrm{id} = L_e.$$

2. Suppose G is a topological group, $A \subset G$, $a \in A$, (X, d) metric space and $f\colon A \to X$ a mapping.

a) Prove that f is continuous at a if and only if for every $\varepsilon > 0$ there exists a neighbourhood U of the neutral element $e \in G$ such that
$$d(f(x), f(a)) < \varepsilon$$
for all $x \in A$ which satisfy $xa^{-1} \in U$.

b) Suppose A above is **compact**. Prove that $f\colon A \to X$ is continuous if and only if for every $\varepsilon > 0$ there exists a neighbourhood U of the neutral element $e \in G$ such that
$$d(f(x), f(y)) < \varepsilon$$
for all $x, y \in A$ which satisfy $xy^{-1} \in U$.

3. Suppose G is a compact topological space and let
$$V = \{f\colon G \to \mathbb{R}^n \mid f \text{ is continuous}\}$$
be as above. Define a norm $|\cdot|$ in the vector space V by
$$|f| = \sup\{f(x) \mid x \in G\}.$$
We consider V a metric space with metric $d(f, g) = |f - g|$ defined by this norm.

a) Suppose $f \in V$ and $\varepsilon > 0$. Prove that there exists a a neighbourhood U of the neutral element $e \in G$ such that

$$d(R_g(f), R_h(f)) < \varepsilon$$

for all $g, h \in G$ that satisfy $gh^{-1} \in U$. (Hint: previous exercise.)

b) Prove that $\Phi\colon G \times V \to V$, $\Phi(g, f) = R_g(f)$ is a continuous action of G on V.

4. Suppose G is a compact topological groups and assume there exists unique **right-invariant** integral on G i.e. there exists a unique linear mapping $I\colon \mathrm{Map}(G, \mathbb{R}) \to \mathbb{R}$ such that
   1) $I(1) = 1$,
   2) $I(f) \geq 0$ if $f\colon G \to \mathbb{R}$ is such that $f(g) \geq 0$ for all $g \in G$ and
   3) $I(R_h(f)) = I(f)$ for all $f \in \mathrm{Map}(G, \mathbb{R}), h \in G$.

   Prove that I is also **left-invariant** i.e.

   $$I(L_h(f)) = I(f)$$

   for all $f \in \mathrm{Map}(G, \mathbb{R}), h \in G$. (Hint: uniqueness of I.)

5. Suppose G is a finite group (with discrete topology). Prove the existence and uniqueness of Haar integral for G. Give a precise formula for the Haar integral of continuous $f\colon G \to \mathbb{R}$.

6. Suppose G is a compact group and assume the existence and uniqueness of Haar integral on G. Let $f\colon G \to \mathbb{R}$ be continuous. Prove that

   $$\int_G f(g)dg = \int_G f(g^{-1})dg.$$

   (Hint: uniqueness of Haar integral.)

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.