Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 6
28.02-02.03.2012

1. Suppose G is a compact topological group and H its closed subgroup. Let $X = G/H$ be a G-space with standard action $g \cdot g'H = gg'H$.
Prove that $X^H = N(H)/H$. Then show that for every $g \in G$ if $gX^H \subset X^H$, then $g \in N(H)$.
Why we need to assume that G is compact?

Solution: Suppose $x = gH$ $g \in G$. Then $x \in X^H$ if and only if $hgH = gH$ for all $h \in H$ if and only if $g^{-1}hg \in H$ for all $h \in H$. This is equivalent to $g^{-1}Hg \subset H$. By Lemma 1.16, since G is compact, this is equivalent to $g \in N(H)$. Hence $X^H = N(H)/H$.

Suppose $g \in G$ and $gX^H \subset X^H$. Since $H \in X^H = N(H)/H$, this implies that $gH \in N(H)/H$, so $g \in N(H)$. In particular this shows that the result of the Lemma 1.13 (X^H is $N(H)$-equivariant) is in general "the best possible ".

If G would not be compact we could only say that
$$X^H = \{gH \mid H \subset gHg^{-1}\} \supset N(H)/H$$
and the last inclusion may be genuine, as the next exercise shows.

2. Prove that
$$H = \{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z}\}\}$$
is a closed subgroup of $G = GL(2, \mathbb{R})$ isomorphic (as a topological group) to the group $(\mathbb{Z}, +)$. Let
$$g = \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \in G,$$
where $m \in \mathbb{N}, m > 1$. Prove that gHg^{-1} is a proper subset of H (i.e. $gHg^{-1} \subsetneq H$). Also prove that any non-trivial subgroup of H is of the form gHg^{-1} for some $g \in G$.

Solution: Define a mapping $f \colon \mathbb{Z} \to G$ by
$$f(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$
It is easily seen that f is an embedding i.e. a homeomorphism to the image. It is also a group homomorphism since
$$f(n)f(m) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+m \\ 0 & 1 \end{bmatrix}$$

for all $n, m \in \mathbb{Z}$. In particular $f(\mathbb{Z}) = H$ is a subgroup of G isomorphic to $\mathbb{Z}$ as a topological group. It is also easily seen that H is closed in G. Let

$$g = \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \in G, m \in \mathbb{N}.$$

Direct computation shows that

$$g \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} g^{-1} = \begin{bmatrix} 1 & mn \\ 0 & 1 \end{bmatrix},$$

so $gHg^{-1} = f(m\mathbb{Z})$, which is a proper subgroup of $H = f(\mathbb{Z})$ if $m > 1$. Since any non-trivial subgroup of $\mathbb{Z}$ is of the form $m\mathbb{Z}$ for $m > 0$, it also follows that every non-trivial subgroup of H is f the form gHg^{-1}.

3. Define G as a subset

$$G = \{\begin{bmatrix} 4^n & 4^m k \\ 0 & 1 \end{bmatrix} \mid n, m, k \in \mathbb{Z}\}$$

of $GL(2, \mathbb{R})$. Prove that G is a subgroup of $GL(2, \mathbb{R})$ and H (defined as in exercise 1) is its subgroup. Let

$$g = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \in G$$

and

$$K = \{\begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z}\}\}.$$

Prove that

$$gHg^{-1} \subsetneq K \subsetneq H$$

and K is not a conjugate of H in G.
Conclude that the claim of Lemma 1.16 is not necessarily true in general for non-compact groups.

Solution: Since $4^m \in \mathbb{Z}$ for $m > 0$, it follows that

$$G = \{\begin{bmatrix} 4^n & 4^m k \\ 0 & 1 \end{bmatrix} \mid n, m, k \in \mathbb{Z}, m < 0\} =$$

$$\{\begin{bmatrix} 4^n & \frac{k}{4^m} \\ 0 & 1 \end{bmatrix} \mid n, m, k \in \mathbb{Z}, m \geq 0\}\}.$$

It is straightforward to verify that the set

$$\{\frac{k}{4^m} \mid k \in \mathbb{Z}, m > 0\} \subset \mathbb{R}$$

is closed under addition and multiplication of real numbers. It follows that

$$\begin{bmatrix} 4^n & \frac{k}{4^m} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4^{n'} & \frac{k'}{4^{m'}} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4^{n+n'} & 4^n \frac{k'}{4^{m'}} + \frac{k}{4^m} \\ 0 & 1 \end{bmatrix} \in G,$$

so G is closed under multiplication. It is also closed under inverse, since

$$\begin{bmatrix} 4^n & \frac{k}{4^m} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 4^{-n} & -\frac{k}{4^{n+m}} \\ 0 & 1 \end{bmatrix}.$$

Hence G is a topological group. Choosing $n = 0, k \in \mathbb{Z}, m = 0$ we see that $H \subset G$. Let

$$g = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

The same calculation as in the previous exercise shows that

$$gHg^{-1} = \{\begin{bmatrix} 1 & 4n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z}\},$$

which clearly is a proper subset of K, which is a proper subset of H. It remains to show that K is not a conjugate of H. Suppose

$$g' = \begin{bmatrix} 4^n & 4^m k \\ 0 & 1 \end{bmatrix}$$

is arbitrary element of G. Then

$$g' \begin{bmatrix} 1 & n' \\ 0 & 1 \end{bmatrix} g'^{-1} = \begin{bmatrix} 1 & 4^n \cdot n' \\ 0 & 1 \end{bmatrix},$$

so the group $g'Hg'^{-1}$ either contains H (when $n \geq 0$) or does not contain an element

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

of K, so never equals K. Hence K is not a conjugate of H in the group G. Since K is a subset of conjugate of H (H itself) and H is a subset of the conjugate $g^{-1}Kg$ of K, this provides a counter-example to the claim of Lemma 1.16 for non-compact groups.

4. a) Suppose G is compact group, X is a Hausdorff G-space and H is a closed subgroup of X. Prove that $X^{[H]}$ is closed in X.

   b) Consider the action of $\mathbb{Z}_2 = \{1, -1\}$ on S^1 given by $(-1) \cdot z = \overline{z}$, $z \in S^1$. (Reminder: the complex conjugate of a complex number $z = a + bi$ is $\overline{z} = a - bi$). Calculate the subset $X_{[H]}$ for $H = \{1\}$ trivial subgroup of $\mathbb{Z}_2$ and show that it is not closed in S^1. Also calculate $X_{[K]}$ for $K = \mathbb{Z}_2$ and show that it is closed in S^1. How can the last claim also be deduced from a)? (Hint: can you write $X_{[K]}$ as $X^{[N]}$ for some subgroup N?)

Solution: a)By Lemma II.1.20 $X^{[H]} = GX^H$ and by Lemma II.1.12 X^H is closed in X. Since G is compact the action mapping $\Phi\colon G \times X \to X$ is closed (Theorem I.1.9), so $GH^H = \Phi(G \times X^H)$ is closed.

b)

$$X_{[\{1\}]} = \{x \in S^1 \mid [G_x] = [\{1\}]\} = \{x \in S^1 \mid G_x = \{1\}\} = S^1 \setminus \{1, -1\},$$

which is clearly not closed in S^1. Similarly $X_{[\mathbb{Z}_2]} = \{1, -1\}$, which is closed. Now

$$X_{[\mathbb{Z}_2]} = X^{[\mathbb{Z}_2]},$$

(since $\mathbb{Z}_2$ is a maximal subgroup of itself), so this conclusion also follows from a).

5. Suppose $f\colon X \to Y$ is a G-equivariant mapping between G-spaces X and Y. Let H be a subgroup of G. Prove that $f(X^H) \subset Y^H$ and $f(X^{[H]}) \subset Y^{[H]}$. Show by examples that $f(X_H)$ is not necessarily a subset of Y_H and $f(X_{[H]})$ is not necessarily a subset of $Y_{[H]}$.

   Solution: Since f is equivariant, we have $G_x \subset G_{f(x)}$ for every $x \in X$ (Lemma 1.24). Hence we obtain immediately
   $$f(X^H) = \{f(x) \mid H \subset G_x\} \subset \{y \in Y \mid H \subset G_{f(x)}\} = Y^H,$$
   $$f(X^{[H]}) = \{f(x) \mid [H] \leq [G_x]\} \subset \{y \in Y \mid [H] \subset [G_{f(x)}]\} = Y^{[H]}.$$
   Let G be any abelian group acting on a space X non-trivially, for instance take action of $\mathbb{Z}_2$ on S^1 from the previous exercise. Define trivial action of G on X/G, then canonical projection $p\colon X \to X/G$ is G-map and there exists at least one point $x \in X$ such that $G_x \neq G_{f(x)} = G$, since action was non-trivial. It follows that for $H = G_x$ $p(X_H) = p(X_{[H]})$ (since G is abelian) is non-empty (contains at least $p(x)$) while $(X/G)_H = (X/G)_{[H]} = \emptyset$, since G acts trivially on X/G.

6. Suppose $f\colon X \to Y$ is a G-equivariant mapping between G-spaces X and Y. Prove that there exists unique induced continuous mapping $f/G\colon X/G \to Y/G$ such that the diagram
   $$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow{\ f/G\ } & Y/G \end{array}$$
   commutes. Here $\pi_X\colon X \to X/G$ and $\pi_Y\colon Y \to Y/G$ are canonical projections.

   Also prove that if f is an open mapping then f/G is an open mapping.

   Solution: Let $x \in X, g \in G$. Then
   $$(\pi_Y \circ f)(gx) = \pi_Y(f(gx)) = \pi_Y(gf(x)) = \pi_Y(f(x)),$$
   so $\pi_Y \circ f$ factors through p_X, i.e. there exists a unique mapping $f/G\colon X/G \to Y/G$ such that
   $$f_G \circ p_X = \pi_Y \circ f.$$
   Since $\pi_Y \circ f$ is continuous and p_X is quotient mapping, f/G is continuous.

   Suppose f is open. Let $V \subset X/G$ be open. Then, since p_X is surjection we see that
   $$f/G(V) = f/G(p_X(p_X^{-1}(V))) = (f_G \circ p_X)(p_X^{-1}(V)) = (\pi_Y \circ f)(p_X^{-1}(V))$$
   is open, since p_Y and f are both open and $p_X^{-1}(V)$ is open in X.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.