Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 6
28.02-02.03.2012

1. Suppose G is a compact topological group and H its closed subgroup. Let $X = G/H$ be a G-space with standard action $g \cdot g'H = gg'H$.
   Prove that $X^H = N(H)/H$. Then show that for every $g \in G$ if $gX^H \subset X^H$, then $g \in N(H)$.
   Why we need to assume that G is compact?

2. Prove that
   $$H = \{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z}\}\}$$
   is a closed subgroup of $G = GL(2, \mathbb{R})$ isomorphic (as a topological group) to the group $(\mathbb{Z}, +)$. Let
   $$g = \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \in G,$$
   where $m \in \mathbb{N}$. Prove that gHg^{-1} is a proper subset of H (i.e. $gHg^{-1} \subsetneq H$). Also prove that any non-trivial subgroup of H is of the form gHg^{-1} for some $g \in G$.

3. Define G as a subset
   $$G = \{\begin{bmatrix} 4^n & 4^m k \\ 0 & 1 \end{bmatrix} \mid n, m, k \in \mathbb{Z}\}\}$$
   of $GL(2, \mathbb{R})$. Prove that G is a subgroup of $GL(2, \mathbb{R})$ and H (defined as in exercise 1) is its subgroup. Let
   $$g = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \in G$$
   and
   $$K = \{\begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z}\}\}.$$
   Prove that
   $$gHg^{-1} \subsetneq K \subsetneq H$$
   and K is not a conjugate of H in G.
   Conclude that the claim of Lemma 1.16 is not necessarily true in general for non-compact groups.

4. a) Suppose G is compact group, X is a Hausdorff G-space and H is a closed subgroup of X. Prove that $X^{[H]}$ is closed in X.

   b) Consider the action of $\mathbb{Z}_2 = \{1, -1\}$ on S^1 given by $(-1) \cdot z = \overline{z}$, $z \in S^1$. (Reminder: the complex conjugate of a complex number $z = a + bi$ is $\overline{z} = a - bi$). Calculate the subset $X_{[H]}$ for $H = \{1\}$ trivial subgroup of $\mathbb{Z}_2$ and show that it is not closed in S^1. Also calculate $X_{[K]}$ for $K = \mathbb{Z}_2$ and show that it is closed in S^1. How does the last claim can also be deduced from a)? (Hint: can you write $X_{[K]}$ as $X^{[N]}$ for some subgroup N?)

5. Suppose $f\colon X \to Y$ is a G-equivariant mapping between G-spaces X and Y. Let H be a subgroup of G. Prove that $f(X^H) \subset Y^H$ and $f(X^{[H]}) \subset Y^{[H]}$. Show by examples that $f(X_H)$ is not necessarily a subset of Y_H and $f(X_{[H]})$ is not necessarily a subset of $Y_{[H]}$.

6. Suppose $f\colon X \to Y$ is a G-equivariant mapping between G-spaces X and Y. Prove that there exists unique induced continuous mapping $f/G\colon X/G \to Y/G$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow{f/G} & Y/G \end{array}$$

commutes. Here $\pi_X\colon X \to X/G$ and $\pi_Y\colon Y \to Y/G$ are canonical projections.

Also prove that if f is an open mapping then f/G is an open mapping.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.