Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 5
Solutions

1. Prove the following claim using nets.
   Suppose C, X are topological spaces and C is compact. Then projection $pr_2 \colon C \times X \to X$ is a closed mapping.

   Solution: Suppose $F \subset C \times X$ is closed. We need to prove that $pr_2(F)$ is closed in X. it is enough to prove that if a net $(x_n)_{n \in N}$ in $pr_2(F)$ converges to $x \in X$, then $x \in pr_2(F)$. Suppose $(x_n)_{n \in N}$ is such a net. For every $n \in N$ we can find $a_n \in C$ and $b_n \in X$ such that $(a_n, b_n) \in F$ and $pr_2(a_n, b_n) = x_n$. Last condition means that simply $b_n = x_n$. In particular we obtain a net a_n in C. Since C is compact, by going to a subnet if necessary we may assumer that a_n converges to $a \in C$. Now (a_n, b_n) is a net in F, which converges to (a, x), so $(a, x) \in F$, since F is closed. It follows that $x = pr_2(a, x) \in pr_2(F)$.

2. Prove that the mapping $\Phi \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(t, (x, y)) = (x + t, y)$ is action of the group $(\mathbb{R}, +) = G$ on the plane $\mathbb{R}^2 = X$. Prove that the orbit space of this action is homeomorphic to $\mathbb{R}$ and the canonical projection $X \mapsto X/G$ is not a closed mapping.

   Solution: Let us first check that Φ is action. Continuity of Φ is obvious. Also

   $$\Phi(0, (x, y)) = (x, y),$$

   $$\Phi(t, \Phi(t', (x, y))) = \Phi(t, (x + t', y)) = (x + (t + t'), y).$$

   If (x, y) and (x', y') are in the same orbit, it implies that $y = y'$. Conversely (x, y) and (x', y) are on the same orbit for every $x, x' \in \mathbb{R}$, since $t(x, y) = (x', y)$ for $t = x' - x$.

   Let $pr_2 \colon \mathbb{R}^2 \to R$, $pr_2(x, y) = y$. Then pr_2 is continuous and open (Topology II) surjection, hence quotient mapping. Also $pr_2(x, y) = pr_2(x', y')$ if and only if $y = y'$ if and only if points (x, y) and (x', y') are in the same orbit. Hence pr_2 induces homeomorphism between X/G and $\mathbb{R}$.

   Previous paragraph implies that up to a homeomorphism the canonical projection $X \mapsto X/G$ is just a projection $pr_2 \colon \mathbb{R}^2 \to R$, which is known to be not closed mapping. For example if one takes

   $$A = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\},$$

   then A is closed, but $pr_2(A) = \mathbb{R} \setminus \{0\}$ is not closed.

3. Suppose G is a topological group and H is a closed subgroup of G. Use nets to prove that the normalizer of H defined by

$$N(H) = \{g \in G \mid gHg^{-1} = H\}$$

is closed in G.
Also prove the following facts:
1) $N(H)$ is a subgroup of G and H is a normal subgroup of $N(H)$.
2) $N(H)$ is the biggest subgroup of G which contains H as a normal subgroup i.e. if $H \leq K \leq G$ and H is normal in K, then $K \subset N(H)$.

Solution: Suppose $g \in \overline{N(H)}$ and let (g_α) be a net in $N(H)$ that converges to g. Then for every $h \in H$ the net $g_\alpha h(g_\alpha)^{-1}$ converges to ghg^{-1} and stays in H. Since H is closed, the limit $ghg^{-1} \in H$ as well. Hence $gHg^{-1} \subset H$. Similarly using the fact that $(g_\alpha)^{-1}h(g_\alpha)$ converges to $g^{-1}hg$ we see that $gHg^{-1} \subset H$. Hence $g \in N(H)$.

Suppose $g, g' \in N(H)$, then

$$(gg')N(H)(gg')^{-1} = g(g'Hg'{-}1)g^{-1} = gHg^{-1} = H,$$

hence $N(H)$ is closed under multiplication. Moreover multiplying $gHg^{-1} = H$ by g^{-1} on the left and g on the right we obtain $H = g^{-1}Hg$, so if $g \in N(H)$ also $g^{-1} \in N(H)$. If $g \in H$, then $gHg^{-1} = H$, since H is a subgroup, so $H \subset N(H)$, in particular $N(H)$ is non-empty. Thus $N(H)$ is a subgroup.

Suppose $K \subset G$ a subgroup such that $H \subset K$. Then H is normal in K if and only if $gHg^{-1} = H$ for all $k \in K$, hence if and only if $K \subset N(H)$. In particular we see that H is closed in $N(H)$.

4. Suppose X is a Hausdorff G-space. Let $J \subset G$ be arbitrary. Prove that

$$X^J = X^{\overline{H}},$$

where H is a subgroup of G generated by J.

This result implies that it is enough to consider fixed point sets of closed subgroups.

Solution: Suppose $x \in= X^{\overline{H}}$. Since $J \subset \overline{H}$, this implies in particular that $jx = x$ for all $j \in J$, so $x \in X^J$.
It remains to prove the converse inclusion

$$X^J \subset X^{\overline{H}}.$$

Let

$$K = \{k \in J \mid kx = x \text{ for all } x \in X^J\}.$$

Then K is a subgroup:
1) $ex = x$ for neutral element $e \in G$ and $x \in X^J$,
2) if $kx = x$ and $k'x = x$ for all $x \in X^J$, then

$$(kk')x = k(k'x) = kx = x,$$

3) if $kx = x$, then

$$x = k^{-1}(kx) = k^{-1}x.$$

Also K is closed since it is intersection of isotropy groups G_x where $x \in X^J$, which are closed by Lemma 1.3. Since $jx = x$ for all $x \in X^J$ by definition, we see that $J \subset K$. Since H is the smallest subgroup of G that contains J, we must have $H \subset K$. Since K is closed this in turn implies that $\overline{H} \subset K$. In other words if $x \in X^J$, then $hx = x$ for all $h \in \overline{H}$, which implies that $x \in X^{\overline{H}}$. Hence $X^J \subset X^{\overline{H}}$.

Remark: Since Lemma 1.3 is true also for T_1 spaces, it is enough to assume X is T_1. Moreover $X^J = X^H$ is true for any G-space X.

5. Suppose G is a topological group, $A \subset G$ is compact and $B \subset G$ is closed. Use nets to prove that the set

$$AB = \{ab \mid a \in A, b \in B\}$$

is closed in G.

Solution: Suppose $(a_n b_n)$ is a net in AB that converges towards $g \in G$. Since A is compact, we may assume that the net (a_n) converges to some $a \in A$. Continuity of algebraic operations then imply that $b_n = (a_n^{-1}(a_n b_n)$ converges to $a^{-1}g$. Since B is closed this implies that $b = a^{-1}g \in B$. Hence $g = ab \in AB$.

6. a) Consider the standard linear action of $GL(n;\mathbb{R})$ on $\mathbb{R}^n$. Prove that for every subset $J \subset GL(n;\mathbb{R})$ the fixed point set $(\mathbb{R}^n)^J$ is a vector subspace of $\mathbb{R}^n$.

b) Consider the standard action of the orthogonal linear group $O(n)$ on S^{n-1}. Prove that for every subset $J \subset O(n)$ the fixed point set $(S^{n-1})^J$ is homeomorphic to S^r for some $r = -1, \ldots, n-1$. Here $S^{-1} = \emptyset$.

Solution: a)Suppose $x, y \in (\mathbb{R}^n)^J$, $\lambda \in \mathbb{R}$. Let $A \in J$. Then

$$A(x + y) = Ax + Ay = x + y,$$

$$A(\lambda x) = \lambda A(x) = \lambda x.$$

Since $A0 = 0$, $0 \in (\mathbb{R}^n)^J$. We have proved that $(\mathbb{R}^n)^J$ is a vector subspaces of $\mathbb{R}^n$.

b) By a) it is enough to prove that for every vector subspace $V \colon \mathbb{R}^n$ the set $V \cap S^{n-1}$ is homeomorphic to S^r for some $r = -1, \ldots, n-1$. Choose an orthonormal basis $v_1, \ldots, v_r$ for V and complete it to the orthonormal basis $v_1, \ldots, v_r, v_{r+1}, \ldots, v_n$ of $\mathbb{R}^n$. Let o be a matrix whose columns are $v_1, \ldots, v_r, v_{r+1}, \ldots, v_n$ in that order. Then $O \in O(n)$ and $O(e_i) = v_i$ for $i = 1, \ldots, r$. Here $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, $e_i = (0, ldots, 0, 1, 0, \ldots, 0)$. Since O preserves norms, O maps $S^r = \{(x_1, \ldots, x_r, 0, 0, \ldots, 0) \in S^{n-1}\}$ precisely onto $V \cap S^{n-1}$. Since O is a homeomorphism (as a linear isomorphism), we obtain the claim.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.