Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 5
20-24.02.2012

1. Prove the following claim using nets.
   Suppose C, X are topological spaces and C is compact. Then projection $pr_2\colon C \times X \to X$ is a closed mapping.

2. Prove that the mapping $\Phi\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(t,(x,y)) = (x+t,y)$ is action of the group $(\mathbb{R},+) = G$ on the plane $\mathbb{R}^2 = X$. Prove that the orbit of this action is homeomorphic to $\mathbb{R}$ and the canonical projection $X \mapsto X/G$ is not a closed mapping.

3. Suppose G is a topological group and H is a closed subgroup of G. Use nets to prove that the normalizer of H defined by
   $$N(H) = \{g \in G \mid gHg^{-1} = H\}$$
   is closed in G.
   Also prove the following facts:
   1) $N(H)$ is a subgroup of G and H is a normal subgroup of $N(H)$.
   2) $N(H)$ is the biggest subgroup of G which contains H as a normal subgroup i.e. if $H \le K \le G$ and H is normal in K, then $K \subset N(H)$

4. Suppose X is a Hausdorff G-space. Let $J \subset G$ be arbitrary. Prove that
   $$X^J = X^{\overline{H}},$$
   where H is a subgroup of G generated by J.

   This result implies that it is enough to consider fixed point sets of closed subgroups.

5. Suppose G is a topological group, $A \subset G$ is compact and $B \subset G$ is closed. Use nets to prove that the set
   $$AB = \{ab \mid a \in A, b \in B\}$$
   is closed in G.

6. a) Consider the standard linear action of $GL(n;\mathbb{R})$ on $\mathbb{R}^n$. Prove that for every subset $J \subset GL(n;\mathbb{R})$ the fixed point set $(\mathbb{R}^n)^J$ is a vector subspace of $\mathbb{R}^n$.

   b) Consider the standard action of the orthogonal linear group $O(n)$ on S^{n-1}. Prove that for every subset $J \subset O(n)$ the fixed point set $(S^{n-1})^J$ is homeomorphic to S^r for some $r = -1, \dots, n-1$. Here $S^{-1} = \emptyset$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.