Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 3
Solutions

1. a) A subset A of a topological space X is called **locally closed** if every point x of A has an open neighbourhood U (in X) such that $V \cap A$ is closed in V. Prove that $A \subset X$ is locally closed if and only if A is open in its closure $\overline{A}$.

   b) Suppose G is a topological group and H its subgroup, which is locally closed in G. Prove that H is a closed subgroup of G. (Hint: use the fact that open subgroup is closed).

   Solution: a) Suppose A is locally closed in X. For every $a \in A$ choose an open neighbourhood V_a of a in X such that $V_a \cap A$ is closed in V_a. Define

$$U = \bigcup_{a \in A} V_a.$$

   Since U is union of open sets, it is open in X. We claim that $A = U \cap \overline{A}$. Clearly $A \subset U \cap \overline{A}$. Conversely suppose $x \in U \cap \overline{A}$. Then there exists $a \in A$ such that $x \in V_a$. If $x \notin A$, then $x \in V_a \setminus A$, which is a compliment of $V_a \cap A$ in V_a. Since the latter set is assumed to b e closed in V_a, it follows that $V_a \setminus A$ is open in V_a. Since V_a is open in X, it follows that $V_a \setminus A$ is actually open in X, i.e. is a neighbourhood of x. But this neighbourhood clearly does not intersect A, which contradicts the fact that $x \in \overline{A}$. Hence $x \in A$ and we are done.

   Now since $A = U \cap \overline{A}$ it follows that A is open in $\overline{A}$ by the definition of relative topology.

   Conversely suppose A is open in $\overline{A}$, which implies that $A = U \cap \overline{A}$ for some open subset U of X. Now

$$U \cap A = U \cap (U \cap \overline{A}) = U \cap \overline{A}$$

   and the set on the right is a closed subset of U, by the definition of relative topology. Hence $U \cap A$ is closed in U. It follows that for every $a \in A$ the set U is an open neighbourhood of a such that $U \cap A$ is closed in U. Thus A is locally closed.

   b) Suppose H is a locally closed subgroup of a topological group G. Then by a) H is open in its closure $\overline{H}$. But $\overline{H}$ is itself a topological group (Lemma 2.2.), hence H is an open subgroup of the topological group $\overline{H}$. By Lemma 6.1 H is closed in $\overline{H}$. This clearly implies that $H = \overline{H}$, hence closed in G.

2. a) Suppose A is a locally compact subspace of a Hausdorff space X. Prove that A is locally closed in X.

   b) Suppose H is a locally compact subgroup of a topological group G. Prove that H is closed in G. Conclude that every discrete subgroup of a topological group is closed.

   Solution: a) Suppose $a \in A$. Since A is locally compact space there exists a neighbourhood V of a in A such that its closure in A is a compact space. By Topology II this relative closure is the set $\overline{V} \cap A$, where $\overline{V}$ is the closure of V in X. Also by the definition of relative topology there exists an open subset U of X such that $V = U \cap A$. Summarizing these observations we can now say that a has an open neighbourhood U in X such that

$$C = \overline{U \cap A} \cap A$$

   is a compact subset of X. Since X is Hausdorff, C is in particular closed in X (Topology II). Now

$$U \cap A = (U \cap A) \cap \overline{U \cap A} = U \cap C,$$

   which is closed in U, since C is closed in X. Hence A is locally closed.

   b) Follows from a) and Exercise 1b). The last claim follows from the fact that every discrete space is locally compact.

3. Suppose G is a compact group and $\phi\colon \mathbb{Z} \to G$ is an injective homomorphism. Prove that ϕ **is not** embedding i.e. homeomorphism to its image $\phi(\mathbb{Z})$. (Hint: otherwise $\phi(\mathbb{Z})$ is a closed discrete subgroup of G.)
   Construct a concrete example of an injective homomorphism $\phi\colon \mathbb{Z} \to G$, where G is compact group.

   Solution: Suppose ϕ is embedding. Then $\phi(\mathbb{Z})$ is homeomorphic to $\mathbb{Z}$, hence discrete. As an image of homomorphism it is also a subgroup of G. Any discrete subgroup is closed by the previous exercise. Hence $\phi(\mathbb{Z})$ is closed in G, thus also compact, since G was compact. It follows that $\phi(\mathbb{Z})$ is an infinite compact discrete space, which is impossible.

   To provide a concrete example it is enough to find a compact group G and an element $x \in G$ of infinite order (i.e. $x^n \neq e$ for any $n \neq 0$). Then $\phi\mathbb{Z} \to G$, $\phi(x) = x^n$ is an injective homomorphism.
   We can take for instance $G = S^1$ and $x = e^{i2\pi t}$, where $t \in I$ is irrational.

4. Suppose G, G' are topological groups and $f\colon G \to G'$ is a homomorphism of groups which is continuous as a mapping between topological spaces. Prove that

$$N = \operatorname{Ker}(f) = \{g \in G \mid f(g) = e'\}$$

(where e' is a neutral element of G') is a closed normal subgroup of G. Prove that the induced homomorphism

$$\tilde{f}\colon G/N \to G'$$

(defined by $\tilde{f}(gN) = f(g)$) is a continuous injective mapping. Is it necessarily a homeomorphism to its image $f(G')$?

Solution: Kernel of f is the inverse image $f^{-1}\{e'\}$ of the singleton $\{e'\}$ under the continuous mapping f. Every singleton in T_1 space is closed, hence also $\mathrm{Ker}(f)$ is closed.

By the general topology any continuous mapping $f\colon X \to Y$ can be factored through the quotient space $X/\sim_f$, where $\sim_f$ is the equivalence relation defined by $x \sim_f y$ if and only if $f(x) = f(y)$. To be precise there exists unique $\overline{f}\colon X/\sim_f \to Y$ such that $f = \overline{f} \circ \pi$ (where $\pi\colon X \to X/\sim_f$ is a natural projection) and this mapping is a continuous injection. It is homeomorphism precisely when f is a quotient mapping (Topology II).

Since now f is a homomorphism, $f(x) = f(y)$ if and only if $x^{-1}y \in \mathrm{Ker}(f)$, i.e. $X/\sim_f$ is exactly the coset space G/N. Hence $\tilde{f}\colon G/N \to G'$ defined by $\tilde{f}(gN) = f(g)$ is a continuous injective homomorphism.
Previous exercise provides us with an example of continuous injective homomorphism which is not embedding.

5. Suppose G, G' are topological groups and $f\colon G \to G'$ is a surjective continuous homomorphism. Prove that f is a quotient mapping if and only if f is an open mapping.

Solution: Open surjective continuous mapping is always a quotient mapping (Topology II).

Conversely suppose f is a quotient mapping. Then by the previous exercise we can factor f as $f = \overline{f}\colon \pi$, where $\overline{f}$ is then a continuous bijection. Since f is quotient $\overline{f}$ is a homeomorphism (Topology II). In particular it is an open mapping. Since canonical projection $\pi\colon G \to G/\operatorname{Ker} f$ is open mapping (Lemma 4.1), f is open as a composition of two open mappings.

6. Suppose G is a connected topological group and N its discrete normal subgroup. Prove that N is **central** in G i.e.

$$xy = yx \text{ for all } x \in G, n \in N.$$

(Hint: consider the mapping $f\colon G \to N$, $f(g) = gng^{-1}$, where $n \in N$.)

Solution: Suppose $n \in N$. Since N is normal $gng^{-1} \in N$ for every $g \in G$. Hence the mapping $f\colon G \to N$, $f(g) = gng^{-1}$ is well-defined. It is evidently continuous. Since G is connected also the image $f(G)$ is connected subset of N. On the other hand N is discrete, hence totally disconnected, so the image $f(G)$ is a singleton. Since $f(e) = n$, $f(G) = \{n\}$. Hence $gng^{-1} = n$ or equivalently

$$gn = ng$$

for every $g \in G$. Since this is true for any $n \in N$, the claim is proved.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.