Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 2
30.01-03.02.2012

1. Suppose $A = (a_{ij})$ is a real $n \times m$ matrix. Recall that its **transpose** A^T is defined as an $m \times n$ matrix with $(A^T)_{ij} = a_{ji}$. Also recall that the standard inner product $\cdot$ in $\mathbb{R}^n$ is defined by
$$x \cdot y = \sum_{i=1}^{n} x_i y_i,$$
$x, y \in \mathbb{R}^n$.
a) Suppose A is an $n \times m$ matrix as above. Prove that
$$Ax \cdot y = x \cdot A^T y$$
for all $x \in \mathbb{R}^m, y \in \mathbb{R}^n$.
b) Prove that the equation above characterises A^T uniquely, i.e. if B is an $m \times n$ matrix such that
$$Ax \cdot y = x \cdot By$$
for all $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, then $B = A^T$.

2. Suppose A is an $n \times n$ matrix. Prove that the following conditions are equivalent:
1) A is orthogonal i.e. $A^T A = I = AA^T$.
2) A preserves standard inner product in $\mathbb{R}^n$ i.e.
$$Ax \cdot Ay = x \cdot y$$
for all $x, y \in \mathbb{R}^n$.
3) A preserves standard norm in $\mathbb{R}^n$ i.e.
$$|Ax| = |x|$$
for all $x \in \mathbb{R}^n$. (Recall that $|x| = \sqrt{x \cdot x}$).
Also prove that if A is orthogonal, then $\det A = \pm 1$.

3. a) Prove that
$$O(n) = \{A \in M(n \times n, \mathbb{R}) \mid A^T A = I = AA^T\}$$
considered as a subgroup of $GL(n, \mathbb{R})$ is a compact topological group (Hint: a subset of $\mathbb{R}^m$ is compact if and only if it is closed and bounded).
b) We have proved in the lectures that
$$SL(n) = \{A \in M(n \times n, \mathbb{R}) \mid \det A = 1\}$$
is a closed subgroup of $GL(n, \mathbb{R})$. Is it compact?

4. Suppose H is a closed subgroup of $(\mathbb{R}, +)$, $H \neq \mathbb{R}$. Prove that there exists $a \in \mathbb{R}$ such that $H = a\mathbb{Z}$, hence H is discrete and isomorphic to $\mathbb{Z}$. (Hint: prove that $a = \min(H \cap \{x \in \mathbb{R} \mid x > 0\})$ exists.)
   Conclude that every subgroup of $(\mathbb{R}, +)$ is either discrete group isomorphic to $\mathbb{Z}$ or dense in $\mathbb{R}$.
5. Suppose α is an irrational number. Prove that the set
   $$\{n + m\alpha \mid n, m \in \mathbb{Z}\}$$
   is dense in $\mathbb{R}$.
6. Suppose H is a normal (in algebraic sence) subgroup of a topological group G. Prove that $\overline{H}$ is also normal.
   (Reminder: subgroup H is normal if $xH = Hx$ for all $x \in G$.)

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.