Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 13
30.04-04.05.2012

1. Suppose X is a G-space and H is a closed subgroup of G. A subset S of X is called *an H-kernel* if there exists a G-mapping $f\colon X \to G/H$ such that $S = f^{-1}(eH)$. Suppose S is an H-kernel in X. Prove the following:
i) S is closed in X.
ii) S is H-equivariant.
iii) $(S|S) = \{g \in G \mid gS \cap S \neq \emptyset\} = H$.
iv) A G-mapping $f'\colon X \to G/H$ such that $S = f'^{-1}(eH)$ is unique.

2. Suppose X is a G-space, where G is compact, H a closed subgroup of G and $S \subset X$ is an H-kernel. Prove that the mapping $\alpha\colon G \underset{H}{\times} S \to X$, $\alpha[g,s] = gs$ is a G-homeomorphism.

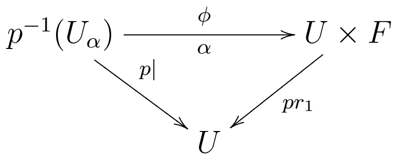
3. Suppose X is a G-space, H a closed subgroup of G and $S \subset X$ is an H-kernel. Denote by $f\colon G \to G/H$ a G-mapping such that $f^{-1}(eH) = S$.
Suppose canonical projection $\pi\colon G \to G/H$ admits a local cross-section $s\colon U \to G$, $U \subset G/H$. Define $\alpha\colon G \underset{H}{\times} S \to X$, $\alpha[g,s] = gs$ as above and let $\beta\colon f^{-1}(U) \to G \underset{H}{\times} S$ be defined by
$$\beta(x) = [s(f(x)), (sf(x))^{-1}x].$$
Show that β is an inverse of the restriction of α on the subset
$$\{[g,s] \mid \ gH \in U\} \subset G \underset{H}{\times} S.$$
Conclude that α is a G-homeomorphism.

4. Suppose X is a G-space, H a closed subgroup of G, $S \subset X$ is an G_x-kernel for some $x \in S$. Prove that $[G_x] \geq [G_y]$ for every $y \in X$.

5. Suppose X is a G-space, H a closed subgroup of G and $S \subset X$ is an H-kernel. Suppose canonical projection $\pi\colon G \to G/H$ admits a local cross-section $s\colon U \to G$, $U \subset G/H$.
Prove that inclusion $i\colon S \hookrightarrow X$ induces a homeomorphism $S/H \to X/G$.

6. Suppose X, E, F are topological spaces and $p\colon E \to X$ is a mapping. We say that p is *locally trivial with fibre F* if there exists an open covering $(U_\alpha)_{\alpha\in\mathcal{A}}$ of X and a homeomorphism $\phi_\alpha\colon p^{-1}U_\alpha \to U_\alpha \times F$ for every $\alpha \in \mathcal{A}$ such that

the diagram



commutes.

a) Prove that p is a continuous and open mapping.

b) Suppose $\alpha, \beta \in \mathcal{A}$ and $x \in U_\alpha \cap U_\beta$. Prove that the *transition function* $\theta^x_{\beta\alpha} \colon F \to F$ defined by

$$\theta^x_{\beta\alpha}(f) = pr_2(\phi_\beta(\phi_\alpha^{-1}(x, f)))$$

is a well-defined homeomorphism.

c) Suppose $\alpha, \beta \in \mathcal{A}$ and $U_\alpha \cap U_\beta \neq \emptyset$. Prove that

$$\phi_\beta \circ \phi_\alpha^{-1}(x, f) = (x, \theta^x_{\beta\alpha}(f))$$

for all $x \in U_\alpha \cap U_\beta, f \in F$.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.