Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 11
Solutions

1. $\mathbb{R}$ acts on $\mathbb{R}^2$ by

$$t \cdot (x, y) = (x + t, y).$$

Prove that $\mathbb{R}^2$ is a Palais proper $\mathbb{R}$-space with this action.

Solution: Suppose $(x, y) \in \mathbb{R}^2$ and let U be any bounded neighbourhood of (x, y). We claim that U is small. Suppose $(z, u) \in \mathbb{R}^2$ any element and let V be a bounded neighbourhood of (z, u). Suppose $t \in \mathbb{R}(U|V)$, then there exists $(a, b) \in V, (c, d) \in U$ such that

$$(a + t, b) = t(a, b) = (c, d)$$

i.e. $b = d, a + t = c$. Since U and V are bounded, there exists a constant $C > 0$ such that $|a|, |c| \leq C$. Triangle inequality then implies, that

$$|t| = |(t + a) - a| \leq |t - a| + |a| = |c| + |a| \leq 2C.$$

Hence $\mathbb{R}(U|V)$ is a bounded subset of $\mathbb{R}$, so its closure is compact.
Hence U is small and space is Palais proper.

2. Suppose G is a compact group, denote $V = Map(G, \mathbb{R})$ and define $\langle , \rangle \colon V \times V \to \mathbb{R}$ by the formula

$$\langle f, g \rangle = \int_G fg.$$

a) Prove that $\langle , \rangle$ is an inner product in V.

We denote the norm induced by this inner product by $\| \cdot \|_2$. In other words

$$\|f\|_2 = \left(\int_G f^2 \right)^{1/2} \text{ for all } f \in V.$$

We also denote

$$\|f\|_1 = \int_G |f| \text{ for all } f \in V.$$

b) Recall that in every inner product so-called Schwartz inequality

$$|\langle a, b \rangle| \leq |a||b|$$

holds (you don't have to prove this). Here $| \cdot |$ on the left is the absolute value of real number and on the right - norm in V induced by $\langle , \rangle$.
Use Schwartz inequality to show that for every $f \in Map(G, \mathbb{R})$

$$\|f\|_1 \leq \|f\|_2.$$

c) Denote by $\| \cdot \|_\infty$ sup-norm in V, i.e.

$$\|f\|_\infty = \sup\{|f(g)| \mid g \in G\}.$$

Prove that $\|f\|_2 \leq \|f\|_\infty$ for all $f \in V$.

Solution: a) Suppose $f, f', g \in V$, $a, b \in \mathbb{R}$.
$\langle , rangle$ is symmetric:

$$\langle f, g \rangle = \int_G fg = \int_G gf = \langle g, f \rangle.$$

Since Haar integral is linear

$$\langle af + bf', g \rangle = \int_G (af + bf')g = \int_G a(fg) + b(f'g)) = a \int_G fg + b \int_G f'g = a\langle f, g \rangle + b\langle f', g \rangle.$$

Together with symmetricity this proves that $\langle , \rangle$ is bilinear. Finally

$$\langle f, f \rangle = \int_G f2 \geq 0$$

and if $f \neq 0$

$$\langle f, f \rangle = \int_G f2 > 0$$

by Proposition 1.5 in Topological Transformation Groups I.

b) We apply Schwartz inequality to f and constant function $1 \in V$ to obtain

$$\|f\|_1 = \int_G |f| = \langle |f|, 1 \rangle \leq \|f\|_2 \|1\|_2 = \|f\|_2.$$

c) Since $|f(g)| \leq \|f\|_\infty$ for all $g \in G$,

$$\|f\|^2 = \int_G f^2 \leq \int_G \|f\|_\infty^2 = \|f\|_\infty^2.$$

Taking square roots from the both sides of the equation yields the claim.

3. Suppose G is a compact group, $V = Map(G, \mathbb{R})$ as above and $\phi \in V$. Define **the convolution operator** $T_\phi \colon V \to V$ by

$$T_\phi(f)(g) = \int_G \phi(gh^{-1}) f(h) fh.$$

a) Prove that T_ϕ is well-defined linear mapping and

$$T_\phi(f)(g) = \int_G \phi(h) f(h^{-1}g) dh.$$

b) Show that

$$\|T_\phi(f)\|_2 \leq \|T_\phi(f)\|_\infty \leq \|\phi\|_\infty \|f\|_1 \leq \|\phi\|_\infty \|f\|_2$$

for all $f \in V$. Conclude that $T_\phi \colon V \to V$ is continuous with respect to $\| \cdot \|_2$ norm (Hint: T_ϕ is linear).

Solution: a) Mapping $G \times G \to \mathbb{R}$ defined by $(h, g) \mapsto \phi(gh^{-1}) f(h)$ is clearly continuous, so by Proposition 1.8 mapping $T_\phi(f)$ is continuous. Hence T_ϕ is well-defined. Linearity of T_ϕ is clear by the linearity of Haar integral. Finally using invariant variable change $h \mapsto h^{-1}g$ we obtain equation

$$T_\phi(f)(g) = \int_G \phi(h) f(h^{-1}g) dh.$$

b)Inequalities $\|T_\phi(f)\|_2 \leq \|T_\phi(f)\|_\infty$ and $\|\phi\|_\infty\|f\|_1 \leq \|\phi\|_\infty\|f\|_2$ follow from the previous exercise. Also for every $g \in G$

$$|T_\phi(f)(g)| = |\int_G \phi(gh^{-1})f(h)fh| \leq \int_G |\phi(gh^{-1})||f(h)|dh \leq \|\phi\|_\infty \int_G |f(h)|dh = \|\phi\|_\infty\|f\|_1.$$

Since T_ϕ is linear, we obtain

$$|T_\phi(f) - T\phi(g)|_2 = |T_\phi(f-g)|_2 \leq \|\phi\|_\infty\|f-g\|_2,$$

which implies that T_ϕ is even Lipschitz with respect to $|\cdot|_2$ norm, hence in particular continuous.

4. Suppose G is a topological group and $\phi\colon G \to GL(\mathbb{R}^n)$ is a continuous linear representation of G in $\mathbb{R}^n$.
Suppose $f\colon G \to \mathbb{R}$ is a matrix coefficient of this representation and H is a *compact* subgroup of G. Show that the mapping $f'\colon G \to \mathbb{R}$ defined by

$$f'(g) = \int_H f(gh)dh$$

is also a matrix coefficient of ϕ, which is constant on the cosets of H, i.e.

$$f'(gh) = f'(g)$$

for all $g \in G, h \in H$.

For the definition of matrix coefficient see exercise 8.5.

Solution: By the definition of the matrix coefficient there exists linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{R}^n$ such that for all $g \in G$

$$f(g) = L(\phi(g)(v)).$$

Now $L \circ \phi(g)$ is a linear mapping $\mathbb{R}^n \to \mathbb{R}$ for all $g \in G$. Since vector-valued Haar integral commutes with linear mappings (Proposition 1.10 in TTG I) and ϕ is a homeomorphism of groups, for every $g \in G$ we obtain

$$f'(g) = \int_H L(\phi(gh)(v))dh = \int_H (L\circ\phi(g))(\phi(h)(v))dh = L\circ\phi(g)(\int_H \phi(h)(v)dh) = L(\phi(g)(v')),$$

where $v' = \int_H \phi(h)(v)dh \in \mathbb{R}^n$. Hence f' is a matrix coefficient by the definition.

The fact that f' is constant on cosets of H is a simple consequence of the invariance of Haar integral:

$$f'(gh) = \int_H f(gh'h)dh' = \int_H f(gh')dh = f'(g).$$

5. Prove the associativity of the twisted product: suppose X is an $G-H$ bispace, Y an $H-K$ bispace and Z is an $K-G'$ bispace. Prove that

$$(X \underset{H}{\times} Y) \underset{K}{\times} Z \cong X \underset{H}{\times} Y \underset{K}{\times} Z \cong X \underset{H}{\times} (Y \underset{K}{\times} Z)$$

as $G-G'$-bispace via the homeomorphisms $[[x,y],z] \mapsto [x,y,z] \mapsto [x,[y,z]]$.

Solution: First we define mapping $(X \times Y) \times Z \to X \underset{H}{\times} Y \underset{K}{\times} Z$ by the formula $(x, y, z) \to [x, y, z]$. This mapping is evidently continuous and factors through $(X \underset{H}{\times} Y) \times Z$, since

$$[xh^{-1}, hy, z] = [x, y, z].$$

Canonical projection $p\colon X \times Y \to X \underset{H}{\times} Y$ is an open surjective mapping, so also $p \times \mathrm{id}\colon (X \times Y) \times Z \to (X \underset{H}{\times} Y) \times Z$ is open surjective, in particular a quotient mapping. Hence the induced mapping $(X \underset{H}{\times} Y) \times Z \to X \underset{H}{\times} Y \underset{K}{\times} Z$ defined by the formula $([x, y], z) \to [x, y, z]$ is continuous.
In the next step we notice that this mapping factors through $(X \underset{H}{\times} Y) \underset{K}{\times} Z$, since

$$([x, y]k^{-1}, kz) \mapsto [x, yk^{-1}, kz] = [x, y, z].$$

Since factorization mapping $q\colon (X \underset{H}{\times} Y) \times Z \to (X \underset{H}{\times} Y) \underset{K}{\times} Z$ is a quotient mapping by definition, the mapping $(X \underset{H}{\times} Y) \underset{K}{\times} Z \to X \underset{H}{\times} Y \underset{K}{\times} Z$ induced by it is continuous. This is the mapping defined by the formula $[[x, y], z] \mapsto [x, y, z]$.
Similarly one constructs the mapping $X \underset{H}{\times} Y \underset{K}{\times} Z \to (X \underset{H}{\times} Y) \times Z$ which is defined by the formula $[x, y, z] \mapsto [[x, y], z]$ and shows that it is continuous.
Two constructed mappings are clearly inverses of each other.

The second part of the claim is proved in the same way.

6. a) Suppose $G = S^1$ and $H = \{1, -1\} = \mathbb{Z}_2$. H acts on $X = [0, 1]$ by

$$(-1) \cdot x = 1 - x.$$

Prove that the space $G \underset{H}{\times} X$ is homeomorphic to the quotient space Y of $S^1 \times I$ with identifications $(x, 0) \sim (-x, 0)$, $x \in S^1$. (Hint: Think of S^1 as I with identifications $0 = 1$. Notice that the restriction of $p\colon G \times X \to G \underset{H}{\times} X$ to $S^1 \times [0, 1/2]$ is a quotient mapping, so $G \underset{H}{\times} X$ is homeomorphic to Y - draw pictures.)

b) Show that Y is homeomorphic to the Mobius band. (Hint: represent Y as a square with identifications. Then cut through the middle and rearrange pieces.) What is S^1-action on Mobius band induced by this homeomorpism?

c) Modify your proofs above to show that $S^1 \underset{H}{\times} S^1$, with action of H on S^1 defined by

$$(-1) \cdot z = \bar{z},$$

is homeomorphic to the Klein's bottle.

Solution: a, b)Let $p\colon S^1 \times X \to S^1 \underset{\mathbb{Z}_2}{\times} X$ be the canonical projection. First of all we notice that

$$p(t, 1 - x) = (t, (-1) \cdot x) = (-t, x),$$

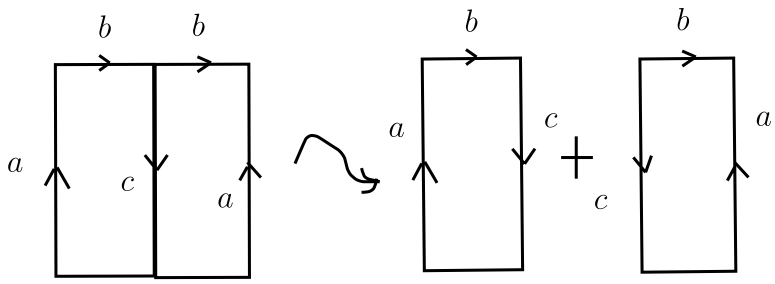
so the restriction of p on the closed subset $S^1 \times [0,1/2]$ is a surjection. Since all spaces involved are compact and Hausdorff (including $S^1 \underset{\mathbb{Z}_2}{\times} X$ which is Hausdorff, since it is an orbit space under action of compact group $\mathbb{Z}_2$), restriction $p|S^1 \times [0,1/2]$ is a closed surjection, hence in particular a quotient mapping. Thus we see that $S^1 \underset{\mathbb{Z}_2}{\times} X$ is homeomorphic to a space induced by $p|S^1 \times [0,1/2]$. Now the only identifications that happen in the set $S^1 \times [0,1/2]$ are the identifications $(t,1/2) \sim (-t,1/2)$.

To visualize situation let us recall that S^1 can be thought of as a quotient space of $I = [0,1]$ where the end points 0 and 1 are identified to a point (for example to a point $(1,0) \in S^1$). A concrete identification can be done through the mapping $x \mapsto e^{2\pi x} = \cos(2\pi x) + i\sin(2\pi x)$. Also we can substitute interval $[0,1/2]$ with the interval $[0,1]$, using homeomorphism $x \mapsto 2x$.

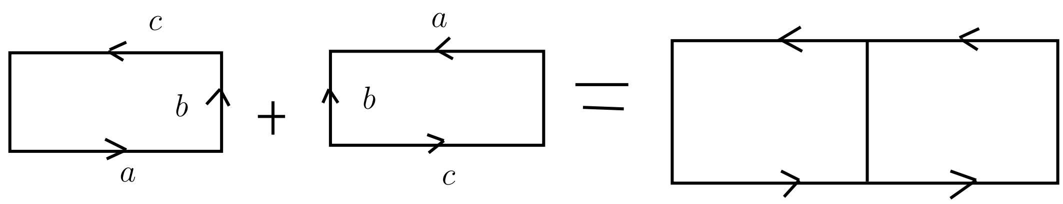
This means that we can think of our space $S^1 \underset{\mathbb{Z}_2}{\times} X$ as a quotient space of the square $I \times I = I^2$ under the identifications $(0,x) \sim (1,x)$ (corresponding to the identification $0 \sim 1$ in S^1, 'a' in the picture below) and $(t,1) \sim (1/2+t,1)$ (corresponding to the identification $(t,1/2) \sim (-t,1/2)$ above, 'b' in the picture below). The action of S^1 on the induced space $S^1 \underset{\mathbb{Z}_2}{\times} X$ in this identification looks like the "horizonal"translation:

$$e2\pi u(x,y) = (x + u(mod1), y).$$

Below is the picture of the square with these two types of identifications marked by the letters 'a' and 'b'. To see that it actually is homeomorphic to Mobius band, we perform the following operations. First we cut the square into two squares using the vertical line $t = 1/2$, which is denoted 'c' in the picture. Now we can think of the space to be obtained from the two squares according to identifications, i.e. "gluings"a', 'b', 'c'.



Next we rearrange the pieces, rotate one of them and glue them back by 'b'. This finally leaves us with the space which is obtained from the square I^2 by identifying $(x,0) \sim (1-x,1)$, which is precisely the Mobius band.



It remains to see how the action of S^1 corresponding to the homeomorphism $S^1 \underset{\mathbb{Z}_2}{\times} X$ to the Mobius band looks like on the Mobius band W. If we trace the original action through all the transformations we made above that this action corresponds to the following action of S^1 on the Mobius band. Suppose (x, y) is a point on the square I^2. We denote the corresponding point on the Mobius band by $\overline{(x, y)}$. Suppose $t \in S^1$ and let $v \in [0, 1[$ be such that $t = e^{2\pi i v}$. Then if $2v + y \leq 1$, i.e. $0 \leq v \leq (1-y)/2$

$$t \cdot \overline{(x, y)} = \overline{(x, y + 2v)}.$$

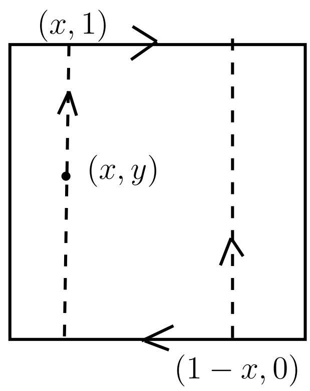
If $(1-y)/2 \leq v \leq (1-y)/2 + 1/2 = 1 - y/2$

$$t \cdot \overline{(x, y)} = \overline{(1 - x, 2v + y - 1)}.$$

Finally if $1 - y/2 \leq v \leq 1$

$$t \cdot \overline{(x, y)} = \overline{(x, 2v + y - 2)}.$$

This can be visualized on the square as following - as v goes from 0 to 1, i.e. t goes around the circle counter-clockwise - point $t \cdot (x, y)$ "travels"up with the constant 2 speed until it reaches the line $y = 1$. Then we jump from the point $(x, 1)$ to the point $(1 - x, 0)$ (which is allowed since we are on the Mobius band) and continus in the same fashion. When we reach the "roof"again we jump back to $(x, 0)$ and continue until we reach (x, y) again.



Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.