Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 11
17-20.03.2012

1. $\mathbb{R}$ acts on $\mathbb{R}^2$ by
$$t \cdot (x, y) = (x + t, y).$$
Prove that $\mathbb{R}^2$ is a Palais proper $\mathbb{R}$-space with this action.

2. Suppose G is a compact group, denote $V = Map(G, \mathbb{R})$ and define $\langle , \rangle \colon V \times V \to \mathbb{R}$ by the formula
$$\langle f, g \rangle = \int_G fg.$$
a) Prove that $\langle , \rangle$ is an inner product in V.

We denote the norm induced by this inner product by $\| \cdot \|_2$. In other words
$$\|f\|_2 = \left(\int_G f^2 \right)^{1/2} \text{ for all } f \in V.$$
We also denote
$$\|f\|_1 = \int_G |f| \text{ for all } f \in V.$$
b) Recall that in every inner product so-called Schwartz inequality
$$|\langle a, b \rangle| \leq |a||b|$$
holds (you don't have to prove this). Here $|\cdot|$ on the left is the absolute value of real number and on the right - norm in V induced by $\langle , \rangle$.
Use Schwartz inequality to show that for every $f \in Map(G, \mathbb{R})$
$$\|f\|_1 \leq \|f\|_2.$$
c) Denote by $\| \cdot \|_\infty$ sup-norm in V, i.e.
$$\|f\|_\infty = \sup\{|f(g)| \mid g \in G\}.$$
Prove that $\|f\|_2 \leq \|f\|_\infty$ for all $f \in V$.

3. Suppose G is a compact group, $V = Map(G, \mathbb{R})$ as above and $\phi \in V$. Define **the convolution operator** $T_\phi \colon V \to V$ by
$$T_\phi(f)(g) = \int_G \phi(gh^{-1}) f(h) fh.$$
a) Prove that T_ϕ is well-defined linear mapping and
$$T_\phi(f)(g) = \int_G \phi(h) f(h^{-1}g) dh.$$
b) Show that
$$\|T_\phi(f)\|_2 \leq \|T_\phi(f)\|_\infty \leq \|\phi\|_\infty \|f\|_1 \leq \|\phi\|_\infty \|f\|_2$$

for all $f \in V$. Conclude that $T_\phi \colon V \to V$ is continuous with respect to $\|\cdot\|_2$ norm (Hint: T_ϕ is linear).

4. Suppose G is a topological group and $\phi \colon G \to GL(\mathbb{R}^n)$ is a continuous linear representation of G in $\mathbb{R}^n$.
Suppose $f \colon G \to \mathbb{R}$ is a matrix coefficient of this representation and H is a *compact* subgroup of G. Show that the mapping $f' \colon G \to \mathbb{R}$ defined by
$$f'(g) = \int_H f(gh)dh$$
is also a matrix coefficient of ϕ, which is constant on the cosets of H, i.e.
$$f'(gh) = f'(g)$$
for all $g \in G, h \in H$.

For the definition of matrix coefficient see exercise 8.5.

5. Prove the associativity of the twisted product: suppose X is an $G-H$ bispace, Y an $H-K$ bispace and Z is an $K-G'$ bispace. Prove that
$$(X \underset{H}{\times} Y) \underset{K}{\times} Z \cong X \underset{H}{\times} Y \underset{K}{\times} Z \cong X \underset{H}{\times} (Y \underset{K}{\times} Z)$$
as $G-G'$-bispace via the homeomorphisms $[[x,y],z] \mapsto [x,y,z] \mapsto [x,[y,z]]$.

6. a) Suppose $G = S^1$ and $H = \{1,-1\} = \mathbb{Z}_2$. H acts on $X = [0,1]$ by
$$(-1)\cdot x = 1 - x.$$
Prove that the space $G \underset{H}{\times} X$ is homeomorphic to the quotient space Y of $S^1 \times I$ with identifications $(x,0) \sim (-x,0)$, $x \in S^1$. (Hint: Think of S^1 as I with identifications $0 = 1$. Notice that the restriction of $p \colon G \times X \to G \underset{H}{\times} X$ to $S^1 \times [0,1/2]$ is a quotient mapping, so $G \underset{H}{\times} X$ is homeomorphic to Y - draw pictures.)

   b) Show that Y is homeomorphic to the Mobius band. (Hint: represent Y as a square with identifications. Then cut through the middle and rearrange pieces.) What is S^1-action on Mobius band induced by this homeomorpism?

   c) Modify your proofs above to show that $S^1 \underset{H}{\times} S^1$, with action of H on S^1 defined by
$$(-1)\cdot z = \bar{z},$$
is homeomorphic to the Klein's bottle.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.