Matematiikan ja tilastotieteen laitos
Transformation Groups
Spring 2012
Exercise 10
02-06.03.2012

1. Suppose G is locally compact and H is a closed subgroup of G. Prove that G/H is Borel proper G-space if and only if H is compact.

2. Consider action of $\mathbb{R}$ on $X = \mathbb{R}^2 \setminus \{0\}$ defined by
$$t(x, y) = (e^t x, e^{-t} y).$$
i) Prove that every point $z \in X$ has a neighbourhood U such that $\overline{G(U|U)}$ is compact (Hint: look at the small ball neighbourhoods).
ii) Let $z = (1, 0)$ and $v = (0, 1)$. Prove that for all neighbourhoods U of z and V of v the set $G(U|V)$ is unbounded, hence not relatively compact.
iii) Prove that X/G is not Hausdorff.

3. Hausdorff space X is called **compactly generated** if the following condition is satisfied:
Suppose $F \subset X$ is such that $F \cap K$ is closed in K for all compact $K \subset X$. Then F is closed.
a) Prove that X is compactly generated if and only if the following condition is satisfied: Suppose $U \subset X$ is such that $U \cap K$ is open in K for all compact $K \subset X$. Then U is open.
b) Prove that every locally compact Hausdorff space is compactly generated (Hint: use a)).
c) Prove that every first-countable space Hausdorff space, in particular every metric space, is compactly generated. Reminder: space is first-countable if every point has a countable neighbourhood base. In first-countable spaces sequences are sufficient to describe eg. closures. (Hint: use the fact that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to $x \in X$, then the set $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is compact.)

4. Suppose X and Y are Hausdroff spaces, and $f \colon X \to Y$ is proper i.e. $f^{-1}K$ is compact for every compact $K \subset Y$. Assume that Y is compactly generated. Show that f is closed mapping. (Hint: enough to prove that $f(C) \cap K$ is closed for every compact $K \subset Y$ and closed $C \subset X$. But $f(C) \cap K = f|f^{-1}(K)(C \cap f^{-1}(K))$.)

5. Suppose X, Y, Z are Hausdorff spaces, $f \colon X \to Y$ and $h \colon X \to Z$ continuous. Suppose $h \circ f$ is proper. Prove that
i) f is proper.
ii) if f is surjection, then h is proper.

6. Suppose X is a G-space and U, V open subsets of X. Prove that $G(U|V)$ is open in G.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.