Nets

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1 Introduction: Sequences

In the very first basic courses on calculus and analysis in Euclidean spaces one learns the importance and convenience of sequences. Practically every definition, new object or result can be expressed, characterised or proved with the use of sequences - and is often done that way. The same techniques, ideas and results based on sequences work also in more general settings of metric spaces or Banach spaces. Let us recall some typical results of metric space theory which involve sequences. In the following X and Y are metric spaces.

(1) Let $A \subset X$ and $x \in X$. Then $x \in \overline{A}$ if and only if there exists a sequence in the set A, which converges to x in X.

(2) Suppose $f\colon X \to Y$ and $x \in X$. Then f is continuous in the point x if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X, which converges to x, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.
Thus we obtain the following characterization of continuity: $f\colon X \to Y$ is continuous if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to $x \in X$, the corresponding sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

(3) X is compact if and only if every sequence in X has a cluster point.

(4) Y is compact if and only if every sequence in Y has a convergent subsequence.

These and many other results may lead one to think that sequences play some universal role in mathematics and that everything can be solved with the use of sequences. This is a generally occurring, but dangerous and false misthought, which might lead to mistakes or incomplete proofs in settings such as the theory of general topological spaces. In fact none of the four results above are true for topological spaces in general. The first two of them do remain true if we restrict ourselves to so-called first countable spaces i.e. spaces in which every point has a countable neighbourhood basis. But even in first countable spaces the results 3 and 4 fail to hold. For example there exists Hausdorff first-countable space in which every sequence has a convergent subsequence, but the space itself is not even paracompact, let alone compact. If the space X is compact, then every sequence in X does have a cluster point, but it need not have convergent subsequence, unless the space is first countable.

In the following we show by explicit counter-examples that the four properties listed above do not hold for topological spaces in general.

Example 1.1. *Let $I = [0,1]$ be the closed unit interval and consider the space $X = I^I$ with product topology. Then X is a separable, connected and compact Hausdorff space.*
Recall that the set X consists of all functions $f\colon I \to I$. Let $A \subset X$ be the set of all $f\colon I \to \{0,1\}$, for which $f^{-1}\{0\}$ is finite and let $f_0 \in X$ be the zero function. Then it is not difficult to see that $f_0 \in \overline{A}$. However if $(f_n)_{n\in N}$ is a sequence in A, then the set

$$\{x \in I : f_n(x) = 0 \text{ for some } n \in \mathbb{N}\}$$

is countable, hence there exists $x \in I$ such that $f_n(x) = 1$ for all $n \in \mathbb{N}$. It follows that (f_n) does not converge to f_0 in X , since that would be equivalent to

$$\lim_{n\to\infty} f_n(x) = 0 = f_0(x)$$

for all $x \in I$.

Example 1.2. *Let X be as above and define the sequence $(f_n)_{n\in\mathbb{N}}$ in X as following: Let $n \in \mathbb{N}$ and $x \in I$. The number x has a unique binary representation of the form*

$$x = \alpha_0, \alpha_1\alpha_2 \ldots \alpha_n \ldots,$$

where $\alpha_m \in \{0,1\}$ for all indices $m \in \mathbb{N}$ and there is no index m_0 such that $\alpha_m = 1$ for all $m \geq m_0$.
Define

$$f_n(x) = \alpha_n.$$

Since X is compact, the sequence f_n has a cluster point. Let us show that this sequence does not have a convergent subsequence. Indeed let $(f_{n_k})_{k\in\mathbb{N}}$ be a subsequence. Define a point $x \in I$ such that

$$x = 0, \alpha_1\alpha_2 \dots \alpha_n \dots,$$

where $\alpha_m = 0$ if $m = n_k$ for some even $k \in \mathbb{N}$ and $\alpha_m = 1$ otherwise. Then the sequence $(f_{n_k}(x))_{k\in\mathbb{N}}$ contains infinitely many 0's and 1's; hence it does not converge. It follows that $(f_{n_k})_{k\in\mathbb{N}}$ does not converge.
This examples shows both that a sequence in a compact space need not to have a convergent subsequence, and that if a sequence has a cluster point then it does not necessarily have a subsequence which converges to this point.

Example 1.3. *Let X and $(f_n)_{n\in\mathbb{N}}$ be as in example 1.2. Let $Y = \overline{\{f_n : n \in \mathbb{N}\}}$. Then Y is a separable compact space. One can prove that in this space only essentially constant sequences converge (see for example [A, Ch. 6]). Here by essentially constant sequence we mean a sequence $(x_n)_{n\in\mathbb{N}}$ that is constant starting from some index $n_0 \in \mathbb{N}$.*
Let Z is any topological space. It follows then, that any function $f : Y \to Z$ has the following property: if a sequence $(y_n)_{n\in\mathbb{N}}$ in Y converges to y, then also the sequence $(f(y_n))_{n\in\mathbb{N}}$ converges to $f(y)$. However since Y is not discrete (it is an infinite compact space) such function may as well be not continuous at any point.

Example 1.4. *Let $(Y, \leq)$ be a well-ordered set, which has the following properties:*
1) Y is uncountable.
2) For every $y \in Y$ the set $\{x \in Y : x \leq y\}$ is countable.
The existence of this set follows easily from well-ordering theorem. In literature Y is usually called "the first uncountable ordinal ". Define the order topology in Y by taking open intervals as a base. Then it can be proved that every sequence in Y has a convergent subsequence. For this we only need to notice that every sequence has a monotonic subsequence (this is true in every ordered set). Now if sequence is decreasing, it has to be essentially constant, since Y is well-ordered, hence it converges. If the sequence is increasing it follows from property 2), that it has a supremum in Y, which will be its limit. However Y is not compact, and not even Lindelöf, since the covering of Y consisting of all bounded open intervals does not have a countable subcovering (this also follows from property 2)). One can show that Y is not even paracompact.

This and many others examples illustrate well enough that in general topological spaces sequences fail to describe the topological nature and properties of the space in the same way they do for metric spaces. This is not surprising at all - after all the image of a sequence is a countable set so if space is "big enough" one cannot expect that one can describe its topological behaviour using only countable subsets.

However since sequences prove to be so convenient in context of, say, metric spaces, it would be helpful to develop some generalised concept which would behave like sequences but work universally in all topological spaces. A concept of nets is exactly such a concept.

2 Nets in topological spaces

The concept of a net is a straightforward generalization of a concept of sequence. Recall that by definition a sequence in a set X is nothing but a mapping $\phi\colon \mathbb{N} \to X$, where $\mathbb{N}$ is a set of natural numbers. It is customary to denote the image point $\phi(n)$ of a sequence ϕ at a point $n \in \mathbb{N}$ by ϕ_n or even x_n. The whole sequence can be referred to as $(\phi_n)_{n\in\mathbb{N}}$ or $(x_n)_{n\in\mathbb{N}}$.

Let X be a topological space and $x \in X$. We say that a sequence $\phi\colon \mathbb{N} \to X$ converges to x if for every neighbourhood U of x, there exists $n_0 = n_0(U)$ such that $x_n = \phi(n) \in U$ for all $n \geq n_0$. A point $x \in X$ is called a cluster point of a sequence $(x_n)_{n\in\mathbb{N}}$ if, for every neighbourhood U of x and every $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ such that $x_n \in U$.

We see that the definition of the convergence of a sequence is expressed not only in terms of topology of a space, but also in terms of a relation $\geq$ in $\mathbb{N}$. In other words we "exploit" the order structure of the set of natural numbers. The idea is that we have to "make a choice" of index n_0 such that beyond that index some property always holds. Sometimes we have to make finitely many choices of indexes. For example recall how we prove that in a Hausdorff space sequence cannot converge towards two different points. We make a counter-assumption that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to different points $x, y \in X$. Then we take neighbourhoods U and V of x and y which do not intersect. By the definition of convergence there exist $n_0, n_1 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$ and $x_n \in V$ for all $n \geq n_1$. Notice that we made two choices of possibly different indexes here. But if $n = \max\{n_0, n_1\}$, then $x_n \in U$ and $x_n \in V$, which is impossible, since $U \cap V = \emptyset$. Notice that the last part of the proof succeeded due to the fact that in $\mathbb{N}$ for every two numbers n_0, n_1 there exists a number n such that $n \geq n_0$ and $n \geq n_1$.

We are now ready to present a generalization of sequence, based on the discussion above. Recall that a binary relation $\leq$ in a set A is, by definition, just a subset of the set $A \times A$. If $(x, y) \in \leq$ we express this by symbols $x \leq y$ and $y \geq x$.

Definition 2.1. *A pair $(N, \leq)$ is called a directed set if $\leq$ is a binary relation in a non-empty set N such that*
1) $n \leq n$ for every $n \in N$ ($\leq$ is reflexive).
2) if $n \leq m$ and $m \leq p$ then $n \leq p$ ($\leq$ is transitive).
3) for every two elements n, m in N there exists an element $p \in N$ such that $n \leq p$ and $m \leq p$ (succesor of both m and n).
We often abuse the notation and say that N is a directed set directed by a relation $\leq$.
If N is a directed set and X is any set, then any mapping $\phi\colon N \to X$ is called a net in X.

We will sometimes denote the net $\phi\colon N \to X$ by the symbols $(\phi_n)_{n\in N}$, $(x_n)_{n\in N}$ or just (x_n).

Example 2.2. *Every ordered set N is a directed set, since for any $n, m \in N$ either $n \leq m$ or $m \leq n$. In particular the set of natural numbers with its usual order is a directed set, so every sequence in a set X is also a net in X. Notice that on the contrary the relation $\leq$ that directs a set need not to be an order or even a partial order, since we don't require antisymmetry: in a directed set it may very well happen that $n \leq m$ and $m \leq n$ for different n and m. For example the trivial binary relation $N \times N$ directs a set N.*

Example 2.3. *Suppose X is any set. Pairs $(\mathcal{P}(X), \subset)$ and $(\mathcal{P}(X), \supset)$ are directed sets, since for any two subsets A and B of X sets $A \cup B$ and $A \cap B$ are also subsets of X.*

Example 2.4. *Let X be any set and let N be the collection of all finite (countable) subsets of X with relation $A \leq B$ meaning $A \subset B$. Then N is a directed set, since for any finite (countable) sets A, B it holds that*

$$A \subset A \cup B,$$

$$B \subset A \cup B,$$

where $A \cup B$ is a finite (countable) subset of X.

Suppose G is an abelian group, I is any set and N is the directed set of all finite subsets of I. Suppose we are given a mapping $i \mapsto x_i$ from the set

I into the group G. Then the mapping $\phi\colon N \to G$,

$$\phi(J) = \sum_{j \in J} x_j$$

is a net in group G.

Example 2.5. *A subset A of the set N directed by* $\leq$ *need not to be directed by restriction of* $\leq$ *on* $A \times A$*, since it may very well happen that every successor of some elments* $m, n \in A$ *is not in A. However for a certain class of subsets of N this problem does not arise.*
We say that a subset M of the set N directed by $\leq$ *is cofinal if for every* $n \in N$ *there exists* $m \in M$ *such that*

$$n \leq m.$$

If M is cofinal, then $(M, \leq |(M \times M))$ *is a directed set.*

Example 2.6. *Suppose* $(N_i, \leq_i)_{i \in I}$ *is a family of directed sets. Then we can direct the product set*

$$N = \prod_{i \in I} N_i$$

in a following canonical way. For $n, m \in N$ *we define* $n \leq m$ *if and only if*

$$n_i \leq_i m_i \text{ for all } i \in I.$$

Reader can easily prove that this indeed defines the structure of directed set in N. Directed set $(N, \leq)$ *is called the product of directed sets* $(N_i, \leq_i)_{i \in N}$.

Example 2.7. *Let X be a topological space and* $x \in X$*. Then the set of all neighbourhoods of x is a directed set N directed by the relation* $\supset$,

$$U \leq V \text{ if and only if } V \subset U.$$

Subset $\mathcal{U}$ *of N is a cofinal subset of N if and only if* $\mathcal{U}$ *is a neighbourhood basic of the point x.*

If we choose a point $x_U \in U$ *for every* $U \in \mathcal{U}$ *we obtain a net* $(x_U)_{U \in \mathcal{U}}$ *in X*

Suppose that N is a directed set and that $\phi\colon N \to X$ is a net. For every $n \in N$ we denote by N_n the set

$$\{m \in N : m \geq n\}.$$

This set is called the tail of N determined by the element $n \in N$. The set $\phi(N_n)$ is called the tail of the net ϕ determined by n. A tail is always nonempty since it contains at least an element n or $\phi(n)$. Notice that by induction and transitivity we obtain that for any finite number of elements $n_1, \dots, n_m$ in N there exists an element $p \in N$ such that $n_i \leq p$ for all $i = 1, \dots, m$. Hence it follows that

$$N_p \subset \cap_{i=1}^m N_{n_i},$$

$$\phi(N_p) \subset \cap_{i=1}^m \phi(N_{n_i}), \tag{2.8}$$

so in particular a finite intersection of tails is always nonempty.

Let X be a topological space and let $\phi\colon N \to X$ be a net in X. We say that ϕ converges to a point $x \in X$ if for every neighbourhood U of x there exists $n_0 \in N$ such that $\phi(n) \in U$ for all $n \geq n_0$. In this case we also say that x is a limit(point) of the net ϕ and denote this by

$$x = \lim \phi = \lim_{n \in N} x_n.$$

We will shortly see that in Hausdorff spaces the limit of a net is unique. However in general a net can have several different limits, so in non-Hausdorff spaces the symbol "=" in a formula

$$x = \lim \phi$$

is not an an equation, but merely just a symbolic way to express the statement " x is a limit point of ϕ ".

A point $x \in X$ is called a cluster point of a net $(x_n)_{n \in N}$ if for every neighbourhood U of x and every $n_0 \in N$ there exists $n \geq n_0$ such that $x_n \in U$. Intuitively this means that a net ϕ " keeps coming arbitrary close to x ". A limit point of a net is also its cluster point but the converse is not true.

The concepts of convergence and cluster point can be easily expressed in terms of tails of a net. By definition $x \in X$ is a limit of a net $\phi : N \to X$ if and only if for every neighbourhood U of x there exists $n \in N$ such that

$$\phi(N_n) \subset U.$$

A point $x \in X$ is a cluster point of ϕ if and only if every neighbourhood U of x interesects with the image of every tail of N, i.e for every $n \in N$

$$U \cap \phi(N_n) \neq \emptyset.$$

In other words it means presicely that

$$x \in \cap_{n \in N} \overline{\phi(N_n)}.$$

It follows that the set of all clusterpoints of a net is closed in X.
Notice that if $N = \mathbb{N}$ this definitions give the same result as traditional definitions for sequences.

Example 2.9. *Let $(\mathcal{U}, \supset)$ be the directed set consisting of a neighbourhood basis of a point $x \in X$. For every $U \in \mathcal{U}$ choose a point $x_U \in U$. Then the net $(x_U)_{U \in \mathcal{U}}$ converges to x.*

Example 2.10. *Let I be a set, $G = (G, +)$ an abelian topological group and $i \mapsto x_i$ a mapping from I to G. In example 2.4 we have defined a net $\phi \colon N \to G$ (where N is a directed set of all finite subsets of I) by*

$$\phi(J) = \sum_{j \in J} x_j.$$

If this net converges we call its limit an (unordered) sum of elements $(x_i)_{i \in I}$ and denote it by

$$\sum_{i \in I} x_i.$$

If $I = \mathbb{N}$ we have another notion of such a sum as series

$$\sum_{i=0}^{\infty} x_n = \lim_{n \to \infty} \sum_{i=0}^{n} x_i$$

based on the limit of sequence. These notions are NOT equivalent. If the unordered sum

$$\sum_{n \in \mathbb{N}} x_i = S$$

exists, then also the series $\sum_{i=0}^{\infty} x_i$ converges to the same point S, but the converse does not hold - a series may converge, although the corresponding unordered sum does not exist. In fact one can prove that an unordered sum $\sum_{n \in \mathbb{N}} x_i$ exists if and only if the series

$$\sum_{i=0}^{\infty} x_{\delta(i)}$$

converges to the same limit for all bijections $\delta \colon \mathbb{N} \to \mathbb{N}$, hence the name unordered sum.
Remark: *Historically the study of unordered summation led Moore into developing the theory of nets.*

Example 2.11. *Let $[a, b] \subset \mathbb{R}$ be a closed interval. Recall that by definition a subdivision S of this interval is a finite ordered set of its points $S = \{a_0, a_1, \ldots, a_n\}$, where $a = a_0 < a_1 < \ldots < a_n = b$. A function c is called a choice function for a subdivision S if it is defined on the set of all intervals $[a_{i-1}, a_i], i = 1, \ldots, n$ and $c([a_{i-1}, a_i]) \in [a_{i-1}, a_i]$ for all $i = 1, \ldots, n$. Consider a set N, which consists of pairs (S, c), where S is a subdivision of of $[a, b]$ and c is a choice function of S. Direct N by the relation $\leq$ defined as follows:*

$$(S, c) \leq (S', c') \text{ iff } S \subset S'.$$

Notice that this is an example of a directed set, which does not satisfy antisymmetry condition.
Suppose $f\colon [a, b] \to \mathbb{R}$ is a mapping. We can define a net $\phi\colon N \to \mathbb{R}$ by asserting

$$\phi(S, c) = \sum_{i=1}^{n} (a_{i-1} - a_i) f(c([a_{i-1}, a_i])).$$

If this net converges we say that f is Riemann integrable and we call its limit the Riemann integral of f, denoted by

$$\int_R f.$$

Next we will show that nets do satisfy the properties we desire.

Proposition 2.12. *Let $A \subset X$ and $x \in X$. Then $x \in \overline{A}$ if and only if there exists a net in A, which converges to x (in X).*

Proof. Suppose $x \in X$ is a cluster point of a net $\phi\colon N \to A$. Then every tail $\phi(N_n)$ of this net is a subset of A, hence

$$x \in \bigcap_{n \in N} \overline{\phi(N_n)} \subset \overline{A}.$$

In other words every cluster point, and hence especially every limit-point of a net in A is in the closure of A.
Let $x \in \overline{A}$. Consider some neighbourhood basis $\mathcal{U}$ of x as a directed set directed by $\supset$ relation. By assumption we can define a net $\phi\colon \mathcal{U} \to A$ such that $\phi(C) \in C$ for every $C \in \mathcal{U}$. It is not difficult to see that ϕ is a net in A that converges towards x. □

Since topology can be completely described in terms of closure, the previous proposition means that the topology of a space can be completely described in terms of the convergence of nets in this space.

Proposition 2.13. *Suppose* $f: X \to Y$ *and* $x \in X$. *Then* f *is continuous at* x *if and only if for every net* $\phi: N \to X$ *which converges to* x, *the net* $f \circ \phi: N \to Y$ *converges to* $f(x)$.

Proof. Suppose f is continuous at x and ϕ is a net that converges to x. Let V be a neighbourhood of $f(x)$. Then there exists a neighbourhood U of x, such that

$$f(U) \subset V.$$

Since ϕ converges to x there exists $n_0 \in N$, such that $\phi(n) \in U$ for all $n \geq n_0$. It follows that for $n \geq n_0$

$$f \circ \phi(n) = f(\phi(n)) \in f(U) \subset V.$$

Hence $f \circ \phi$ converges to $f(x)$.

Suppose conversely the condition is true. It is enough to show that if $x \in \overline{A}$, then $f(x) \in \overline{fA}$. If $x \in \overline{A}$, by the previous composition there exists a net $\phi: N \to A$ such that ϕ converges to x. By the assumption $f \circ \phi$ (which is a net in fA) converges to $f(x)$. Previous proposition now implies that $f(x) \in \overline{fA}$. □

Proposition 2.14. *A net* $\phi: N \to X$ *in a product space* $X = \prod_{i \in I} X_i$ *converges to* $x = (x_i)_{i \in I} \in X$ *if and only if for every* $i \in I$*the net* $\phi_i = pr_i \circ \phi: N \to X_i$ *converges to* $x_i \in X_i$.

Proof. Since the projection maps pr_i are continuous, the "only if " part follows from the previous proposition.

To prove the other direction suppose $\phi: N \to X$ is a net in X and assume that the nets $\phi_i = pr_i \circ \phi$ converge to x_i for every $i \in I$. Let $U = \prod_{i \in I} U_i$ be a canonical neighbourhood of x and let $J \subset I$ be a finite subset of I such that $U_i = X_i$ when $i \notin J$. For every $j \in J$ the set U_j is a neighbourhood of x_i in X_i, so there exists $n_j \in N$, such that

$$pr_j(\phi(n)) \in U_j$$

for every $n \geq n_j$. Since N is a directed set and J is finite, there exists $n_0 \in N$ such that $n_0 \geq n_j$ for every $j \in J$. It follows that for every $n \geq n_0$ and every index $i \in I$ we have

$$pr_i(\phi(n)) \in U_i,$$

hence $\phi(n) \in U$. This shows that ϕ converges to x. □

Proposition 2.15. *Suppose X is Hausdorff. Then every net in X has at most one limit point.*

Proof. Suppose that X is Hausdorff, $\phi\colon N \to X$ is a net in X and $a, b \in X$ are two different points. Then a and b have disjoint neighbourhoods U and V. If ϕ converges to both a and b, then there are $n_0 \in N$ and $n_1 \in N$ such that for every $n \geq n_0$ we have $\phi(n) \in U$ and for every $n \geq n_1$ we have $\phi(n) \in V$. But there exists $n \in N$ such that both $n \geq n_0$ and $n \geq n_1$, so $\phi(n) \in U \cap V = \emptyset$, which is a contradiction.

□

Proposition 2.16. *A topological space X is compact if and only if every net in X has a cluster point.*

Proof. Let X be compact. Suppose $\phi\colon N \to X$ is a net. Consider the family

$$\mathcal{F} = \{\overline{\phi(N_n)} : n \in N\}$$

of closed subsets of X. By (2.8), this family has finite intersection property. Since X is compact it follows that

$$\bigcap_{n \in N} \overline{\phi(N_n)} = \bigcap \mathcal{F} \neq \emptyset.$$

This means that ϕ has a cluster point.

On the contrary suppose that every net in X has a cluster point. Let $\mathcal{F}$ be a family of closed subsets of X which has the finite intersection property. It is enough to show that

$$\bigcap \mathcal{F} \neq \emptyset.$$

Let N be the collection of all finite intersections of elements of the family $\mathcal{F}$, then relation $\supset$ directs N. Since every element of N is non-empty, there exists a net $\phi\colon N \to X$, that has property

$$\phi(F) \in F \text{ for all } F \in N.$$

By assumption this net has a cluster point $x \in X$. Hence x is in the closure of every set in N, especially in the closure of every set $F \in \mathcal{F} \subset N$. But all sets $F \in \mathcal{F}$ are closed, so

$$x \in \bigcap \mathcal{F},$$

and the claim is proved. □

We know that in metric spaces the previous proposition can also be stated in the form "X is compact if and only if every sequence has a convergent subsequence". In order to obtain a similar result for nets we need to generalize the notion of subsequence. This time, however, we cannot make a straightforward generalization.

Recall that by a subsequence of a sequence $(x_n)_{n\in\mathbb{N}}$ one usually means a sequence $(x_{n_k})_{k\in\mathbb{N}}$, where the function $k \mapsto n_k$ is assumed to be a strictly increasing mapping of $\mathbb{N}$ into itself (as we will see soon this is an unnecessarily strong condition, but it is enough for sequences). We could, of course, adopt the same definition for nets and say that a net $\psi\colon N \to X$ is a subnet of a net $\phi\colon N \to X$ if $\psi = \phi \circ \alpha$, where $\alpha\colon N \to N$ is a strictly increasing mapping. But this will not work for our purposes at all, because then subnets of sequences will be subsequences and, as the examples in the first chapter show, the proposition "X is compact if its every net has a convergent subnet" will not be true. So we have to be more clever than that and come up with something more general. Notice that above we didn't change the domain of the net, when trying to define a subnet. If we allow the domain to change, then our goal will be achieved.

Definition 2.17. *Let M and N be directed sets. A mapping $\alpha\colon M \to N$ is called cofinal if for every $n_0 \in N$ there exists $m_0 \in M$ such that $\alpha(m) \geq n_0$ for all $m \geq m_0$.*

Remark: A mapping $\alpha\colon M \to N$ between directed sets is cofinal if and only if the image of every cofinal subset of M is cofinal in N, hence the term " cofinal mapping".

The concept of cofinal mapping reflects the intuitive idea of "m growing arbitrary large as n grows arbitrary large".

Definition 2.18. *Let $\phi\colon N \to X$ be a net in a set X. A net $\psi\colon M \to X$ is called a subnet of ϕ if there exists a cofinal mapping $\alpha\colon M \to N$ such that $\psi = \phi \circ \alpha$.*

A subnet ψ of a sequence ϕ, which is a sequence itself (i.e. the domain of which is $\mathbb{N}$) is called a subsequence of ϕ. Notice that our new definition of sequence includes all subsequences in a traditional sence, but also produce some new subsequences, so our definition of a subsequence is broader than the traditional definition. This, however, does not cause any problem in practice.

Example 2.19. *Itis not difficult to see that a mapping $\alpha\colon \mathbb{N} \to \mathbb{N}$ is cofinal if and only if $\alpha^{-1}(n)$ is finite for every $n \in \mathbb{N}$.*

Example 2.20. *Suppose $(N, \leq)$ is a directed set and $M \subset D$ such that the restriction of $\leq$ on $M \times M$ directs M. Then the canonical inclusion mapping $i\colon M \to N$ is cofinal if and only if M is cofinal. If $\phi\colon N \to X$ is a net and A is a cofinal subset of N, than $\phi|A$ is a subnet of ϕ. Such subnets are called cofinal subnets. Every subsequence of a sequence ϕ is a cofinal subnet of ϕ.*

Proposition 2.21. *A point $x \in X$ is a cluster point of a net ϕ if and only if there exists a subnet of ϕ which converges to x. If ϕ converges to x then its every subnet also converges to x. If x is a cluster point of a subnet of ϕ, then x is also a cluster point of ϕ.*

Proof. Suppose $\psi = \phi \circ \alpha$ is a subnet of ϕ, $\alpha\colon M \to N$ is a cofinal mapping. Suppose, that $x \in X$ is a cluster point of ψ and let $n_0 \in N$, V be a neighbourhood of x in X. Then there exists $m_0 \in M$, such that $\alpha(m) \geq n_0$ for all $m \geq m_0$. Since x is a cluster point of ψ, there exists $m \geq m_0$ such that $\psi(m) = \phi(\alpha(m)) \in V$. Since $n = \alpha(m) \geq n_0$, it follows that x is a cluster point of ϕ as well and the last claim is proved. Also it follows that if x is a limit of ψ, it is also a cluster point of ϕ.
Suppose $x = \lim \phi$ and let U be a neighbourhood of x. There exists $n_0 \in N$ such that $\phi(n) \in U$ for every $n \geq n_0$. Since ψ is a subnet, there exists $m_0 \in M$, such that $\alpha(m) \geq n_0$ for every $m \geq m_0$. It follows that for every $m \geq m_0$

$$\psi(m) = \phi(\alpha(m)) \in U.$$

Hence $x = \lim \psi$.
Suppose x is a cluster point of a net $\phi\colon N \to X$. Let $\mathcal{U}$ be the set of all neighbourhoods of x in X and let $P = \mathcal{U} \times N$ be a product of directed sets $\mathcal{U}$ and N. Consider the subset

$$M = \{(U, n) \in P \mid \phi(n) \in U\}$$

of P. Then M is cofinal in P: suppose $(U, n) \in P$. Since x is a cluster point of ϕ there exists $n' \geq n$ such that $\phi(n') \in U$. Now $(U, n') \in M$ and $(U, n') \geq (U, n)$. Hence M is cofinal in P, in particular M itself is a directed set.
Define $\alpha\colon M \to N$ by $\alpha(U, n) = n$. It is not difficult to see that α is cofinal: let $n_0 \in N$, then for every $(U, n) \geq (X, n_0)$ it is true that

$$\alpha(U, n) = n \geq n_0.$$

Hence $\psi = \phi \circ \alpha$ is a subnet of ϕ. This net converges to x, since for every neighbourhood U of x

$$\psi(V, n) \in V \subset U$$

when $(V, n) \geq (U, n')$, where n' is any fixed element of N. $\square$

Corollary 2.22. *X is compact if and only if every net in X has a convergent subnet.*

Example 2.23. *Let N be the set of pairs (S, c), considered in Example 2.11. Define the length $\|S\|$ of subdivision $S = \{a_0, a_1, \ldots, a_n\}$ to be the greatest of the lenghts of intervals $[a_{i-1}, a_n]$, $i = 1, \ldots, n$. Define relation $\ll$ in N such that*

$$(S, c) \ll (S', c') \text{ iff } \|S'\| \leq \|S\|.$$

Then $(N, \ll)$ will be a directed set. In Example 2.11 we also defined a different relation $\leq$ in N. Now the identity mapping $id\colon (N, \leq) \to (N, \ll)$ is increasing (if $S \subset S'$ then $\|S'\| \leq \|S\|$), hence cofinal. It follows that if $f\colon [a, b] \to \mathbb{R}$ is a mapping, and $\phi : (N, \leq) \to \mathbb{R}$ is a net defined in an Example 2.11, then it is a subnet of the net $\phi' = \phi\colon (N, \ll) \to \mathbb{R}$ defined by the same formula. Hence if ϕ' converges also ϕ converges and f is Riemann-integrable. One can prove that also converse statement is true: if ϕ converges, also ϕ' converges (this is proved in the course Analysis II).

Next we will give an example on a typical use of nets for proving theorems. It is important to notice that we can often use nets formally in exactly the same way as sequences, and a proof done with sequences for a special case often works for the general case if sequences are replaced with nets. Often one can even write a general proof by simply exchanging every word ”sequence” with a word ”net” at its every appearance!

Theorem 2.24. *Suppose G is a compact group acting on a topological space X. Then the action mapping $\Phi\colon G \times X \to X$ is closed.*

Proof. Let C be a closed subset of $G \times X$ and suppose $y \in \overline{\Phi(C)}$. It is enough to show that $y \in \Phi(C)$. Since $y \in \overline{\Phi(C)}$, it follows that there exists a net $\phi\colon N \to \Phi(C)$, which converges to y. For every $n \in N$ there exist $g_n \in G$ and $x_n \in X$ such that $(g_n, x_n) \in C$ for every $n \in N$ and

$$\phi(n) = \Phi(g_n, x_n) = g_n x_n.$$

Hence we can define nets $\phi_1\colon N \to G$, $\phi_2\colon N \to X$ by

$$\phi_1(n) = g_n,$$

$$\phi_2(n) = x_n.$$

Since G is compact, there exists a subnet $\psi_1 = \phi_1 \circ \alpha\colon M \to G$, which converges to a point $g \in G$, here $\alpha\colon M \to N$ is a cofinal mapping. Let $\psi_2 = \phi_2 \circ \alpha\colon M \to X$, then ψ_2 is a subnet of ϕ_2. Now a net $M \to X$ defined by

$$m \mapsto \Phi(\psi_1(m), \psi_2(m))$$

is a subnet of ϕ, hence is a net in $\Phi(C)$ which also converges to y, so we may assume ϕ is such that ϕ_1 converges in G to a point g. Now, since Φ is a continuous action, we obtain that

$$\phi_2(n) = x_n = g_n^{-1} g_n x_n = \Phi(g_n^{-1}, \Phi(g_n, x_n))$$

converges to $\Phi(g^{-1}, y) = g^{-1}y$. It follows that the product net (g_n, x_n) in C converges to a pair $(g, g^{-1}y)$. Since C is closed, also limit point $(g, g^{-1}y)$ is an element of C. Hence

$$y = g(g^{-1}y) = \Phi(g, g^{-1}y) \in \Phi(C)$$

and the claim is proved. □

I advise the reader to submit a proof for the previous proposition assuming that both spaces G and X are first countable spaces and to compare it to the general proof above.

References

[A] Aleksandroff, P.S. *Introduction to the set theory and general topology, second edition* Nauka, Moscow, 1977

[V] Väisälä, Jussi *Topologia I,* Limes ry 1999.