Constructing the Haar integral for compact topological groups

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1 Preliminaries

Lemma 1. Suppose that G is a topological group and M is an arbitrary subspace of G. Then a function $f\colon M \to \mathbb{R}$ is continuous at $a \in M$ if and only if for all $\varepsilon > 0$ there exists a neighbourhood U of identity such that

$$|f(x) - f(a)| < \varepsilon$$

for all $x \in M$ satisfying $xa^{-1} \in U$.

Proof. Exercise 7.2. (a) □

Definition 1. Suppose that G is a topological group and M is a subspace of G. Then a function $f\colon M \to \mathbb{R}$ is said to be **uniformly continuous in the 1st sense** if for all $\varepsilon > 0$ there exists a neighbourhood U of identity so that

$$|f(x) - f(y)| < \varepsilon$$

for all $x, y \in M$ satisfying $xy^{-1} \in U$.

The function f is said to be **uniformly continuous in the 2nd sense** if for all $\varepsilon > 0$ there exists a neighbourhood U of identity so that

$$|f(x) - f(y)| < \varepsilon$$

for all $x, y \in M$ satisfying $x^{-1}y \in U$.

The two definitions above are not in general equivalent. However, we have the following lemma.

Lemma 2. Let G be a topological group and M a compact subspace of G. Then any continuous function $f\colon M \to \mathbb{R}$ is uniformly continuous in both of the senses in the previous definition.

Proof. Exercise 7.2. (b) □

Definition 2. Let G be a topological group, M its subspace and Δ a family of functions $M \to \mathbb{R}$. Then we say that the family Δ is **uniformly equicontinuous** if and only if for every $\varepsilon > 0$ there exists a neighbourhood U of e such that

$$|f(x) - f(y)| < \varepsilon$$

holds for every $f \in \Delta$ whenever $x, y \in M$ satisfy $xy^{-1} \in U$.

Moreover, we say that Δ is **uniformly bounded** if there exists $M \in \mathbb{R}$ such that

$$|f(x)| < M$$

for all $f \in \Delta$, $x \in M$.

Definition 3. Let X be a set and $f_k \colon X \to \mathbb{R}$ $(k = 1, 2, \dots)$ a sequence of functions. We say that f_k converges uniformly to a function $f \colon X \to \mathbb{R}$ if and only if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ so that for every $k \geq n$ we have

$$|f(x) - f_k(x)| < \varepsilon$$

for all $x \in X$.

Lemma 3. *A sequence of functions $f_k \colon X \to \mathbb{R}$ $(k = 1, 2, \dots)$ converges uniformly if and only if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every $p, q \geq n$ we have*

$$|f_p(x) - f_q(x)| < \varepsilon$$

for all $p, q \in \mathbb{N}$.

Proof. Suppose first that f_k converges uniformly to f. Let $\varepsilon > 0$ be given. Then there exists $n \in \mathbb{N}$ so that for every $k \geq n$ we have

$$|f_k(x) - f(x)| < \varepsilon/2$$

for all $x \in X$. Then if $p, q \geq n$, we also have

$$|f_p(x) - f_q(x)| \leq |f_p(x) - f(x)| + |f_q(x) - f(x)| < \varepsilon.$$

For the other direction the condition in the statement of the lemma implies that for any fixed $x \in X$ the sequence $f_k(x)$ $(k \in \mathbb{N})$ is Cauchy. Hence for all $x \in X$ there exists a limit $f(x)$. Now let $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that

$$|f_p(x) - f_q(x)| < \varepsilon/2$$

holds for all $p, q \geq n$, $x \in X$. Now for any fixed $x \in X$ we have

$$|f(x) - f_q(x)| \leq |f(x) - f_p(x)| + |f_p(x) - f_q(x)| < |f(x) - f_p(x)| + \frac{\varepsilon}{2},$$

and choosing large enough p, we get

$$|f(x) - f_q(x)| < \varepsilon.$$

Because this holds for every x and $q \geq n$, we see that the sequence f_k converges uniformly to f. □

Lemma 4. *Suppose that the set X is endowed with topology and $f_k \colon X \to \mathbb{R}$ is a sequence of continuous functions converging uniformly to a function $f \colon X \to \mathbb{R}$. Then f is continuous.*

Proof. Fix $a \in X$. We'll show that f is continuous at a. That is, given $\varepsilon > 0$, we have to find an open neighbourhood U of a such that $|f(x) - f(a)| < \varepsilon$ for every $x \in U$. Because the sequence f_k is uniformly convergent, there exists $n \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon/3$$

for all $x \in X$. Moreover because f_n is continuous, there exists a neighbourhood U of a such that

$$|f_n(x) - f_n(a)| < \varepsilon/3$$

for all $x \in U$. Hence

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < \varepsilon$$

for all $x \in U$. ☐

Lemma 5. Let G be a topological group and M a compact subspace of G. Then any uniformly convergent sequence $f_k \colon M \to \mathbb{R}$ $(k = 1, 2, \dots)$ of continuous functions is uniformly equicontinuous and uniformly bounded.

Proof. Let $\varepsilon > 0$ be given and suppose that f_k converges to f uniformly. Since f is continuous and M is compact, f is also uniformly continuous by Lemma 2. Hence there exists a neighbourhood V of identity such that $|f(x) - f(y)| < \varepsilon/3$ for all $x, y \in M$ satisfying $xy^{-1} \in V$. Moreover there exists $n \in \mathbb{N}$ such that for all $k \geq n$ we have

$$|f(x) - f_k(x)| < \varepsilon/3$$

for all $x \in M$. Thus

$$|f_k(x) - f_k(y)| \leq |f_k(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_k(y)| < \varepsilon$$

whenever $x, y \in M$ satisfy $xy^{-1} \in V$. Now denote by V_i, $i = 1, 2, \dots, n-1$, a neighbourhood of identity such that

$$|f_i(x) - f_i(y)| < \varepsilon$$

for all $x, y \in M$ satisfying $xy^{-1} \in V_i$. If we now let $U = V \cap \bigcap_{i=1}^{n-1} V_i$, then for all $x, y \in M$ satisfying $xy^{-1} \in U$ we have

$$|f_k(x) - f_k(y)| < \varepsilon$$

for all $k \in \mathbb{N}$. Hence the sequence f_k is uniformly equicontinuous.

The uniform boundedness follows from the compactness of M: The functions $f_1, \dots, f_{n-1}$ are bounded by some constant M and the rest of the functions are bounded by $\sup_{x \in M} |f(x)| + \varepsilon$. ☐

Definition 4. Let X be a compact topological space and $f \colon X \to \mathbb{R}$ a continuous function. Then the **oscillation** of f is the real number

$$\operatorname{osc} f = \sup_{x \in X} f(x) - \inf_{x \in X} f(x).$$

Lemma 6. Assume that X is a compact topological space and suppose that $f_k\colon X \to \mathbb{R}$ is a sequence of continuous functions converging uniformly to $f\colon X \to \mathbb{R}$. Then we have

$$\lim_{k\to\infty} \min_{x\in X} f_k(x) = \min_{x\in X} f(x)$$
$$\lim_{k\to\infty} \max_{x\in X} f_k(x) = \max_{x\in X} f(x)$$
$$\lim_{k\to\infty} \operatorname{osc} f_k = \operatorname{osc} f.$$

Proof. Just notice that for any $\varepsilon > 0$ and large enough k we have

$$\min_{x\in X} f(x) - \varepsilon \le \min_{x\in X} f_k(x) \le \min_{x\in X} f(x) + \varepsilon,$$

since if the minimum of f is attained at a point $a \in X$, then

$$\min_{x\in X} f_k(x) \le f_k(a) \le f(a) + \varepsilon$$

and if the minimum of f_k is attained at a point $b \in X$, then

$$f_k(b) \ge f(b) - \varepsilon \ge f(a) - \varepsilon.$$

A similar deduction works for maximum. The result for oscillation follows immediately from the first two. ☐

Finally we'll prove the following version of Arzelà-Ascoli theorem.

Theorem 1. Let G be a topological group and M a compact subspace of G. If Δ is a uniformly equicontinuous and uniformly bounded family of functions $M \to \mathbb{R}$, then every sequence of functions $f_k \in \Delta$ $(k = 1, 2, \dots)$ contains a uniformly convergent subsequence.

Proof. We start with the following lemma.

Lemma 7. Suppose that Δ' is an infinite, uniformly equicontinuous and uniformly bounded family of functions $M \to \mathbb{R}$. Then for any $\varepsilon > 0$ there exists an infinite subfamily Δ'_ε of Δ' such that

$$|f(x) - g(x)| < \varepsilon$$

for all $f, g \in \Delta'_\varepsilon$, $x \in M$.

Proof. For every $a \in M$ there exists by equicontinuity an open neighbourhood U_a of a such that for all $f \in \Delta'$, $x \in U_a$ we have

$$|f(x) - f(a)| < \frac{\varepsilon}{3}.$$

Now these neighbourhoods cover M and hence we can choose from them a finite subcover $U_{a_1}, \dots, U_{a_n}$. We'll now construct the subfamily Δ'_ε in steps.

First note that for any $a \in M$, the set $\{f(a) : f \in \Delta'\}$ is a bounded subset of $\mathbb{R}$. Hence there exists an interval I of length $\varepsilon/3$ such that $f(a) \in I$ for infinitely many $f \in \Delta'$. Denote by Δ'_a the infinite subfamily of Δ' containing these functions. Then for all $f, g \in \Delta'_a$ we have $|f(a) - g(a)| < \frac{\varepsilon}{3}$. Therefore

$$|f(x) - g(x)| \le |f(x) - f(a)| + |f(a) - g(a)| + |g(a) - g(x)| < \varepsilon$$

for all $f, g \in \Delta'_a$ and $x \in U_a$.

Apply the above argument for $a = a_1$ to obtain a subfamily Δ'_{a_1} such that

$$|f(x) - g(x)| < \varepsilon$$

for $x \in U_{a_1}$, $f, g \in \Delta'_{a_1}$. Then apply it for $a = a_2$ and the family Δ'_{a_1} in place of Δ' to obtain a new family Δ'_{a_1,a_2}. Continue like this to get a family $\Delta'_{a_1,\ldots,a_n} = \Delta'_\varepsilon$ for which

$$|f(x) - g(x)| < \varepsilon$$

holds for all $x \in \bigcup_{k=1}^n U_{a_k} = M$. □

Let now $f_k \in \Delta$ ($k = 1, 2, \ldots$) be a sequence of functions from Δ and let $\Delta' = \{f_k : k \in \mathbb{N}\}$. If Δ' is finite, then one of the functions f_k appears infinitely many times in the sequence and hence we can choose $(f_k, f_k, \ldots)$ as the uniformly convergent sequence. So suppose that Δ' is infinite. Now by the lemma above we can find an infinite subset Δ_1 of Δ' for which $|f(x) - g(x)| < 1$ for all $x \in M$, $f, g \in \Delta_1$. Supposing we have already defined Δ_{n-1} for some $n \geq 2$, we may again use the lemma to find an infinite subset $\Delta_n \subset \Delta_{n-1}$ such that $|f(x) - g(x)| < \frac{1}{n}$ for all $x \in M$, $f, g \in \Delta_n$. Now choose a subsequence g_m of f_k by choosing $g_m = f_{k_m}$ from Δ_m, $m = 1, 2, \ldots$ so that $k_{m+1} > k_m$. We now see that g_m converges uniformly, because if $l \geq m$, then

$$|g_l(x) - g_m(x)| < \frac{1}{m}$$

for all $x \in M$ and hence g_m is uniformly Cauchy. □

2 The Haar integral

Definition 5. Let G be a compact topological group and $C(G)$ the space of continuous functions $G \to \mathbb{R}$. A mapping $I \colon C(G) \to \mathbb{R}$ satisfying the following properties is called a Haar integral on G. (It is customary to write $I(f) = \int f(x)\,dx$.)

(H1) $\int cf(x)\,dx = c\int f(x)\,dx$ for all $f \in C(G)$, $c \in \mathbb{R}$

(H2) $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$ for all $f, g \in C(G)$

(H3) For all $f \in C(G)$ that satisfy $f(x) \geq 0$ ($x \in G$) we have $\int f(x)\,dx \geq 0$.

(H4) If a is an arbitrary element of G, then $\int f(xa)\,dx = \int f(x)\,dx$.

(H5) Let $f \in C(G)$ be defined by $f(x) = 1$ for all $x \in G$. Then $\int f(x)\,dx = 1$.

We make some remarks regarding the definition. First of all notice that (H5) is just a matter of normalization: without it we could multiply any mapping I satisfying (H1)–(H4) by a constant $c \geq 0$ to get another mapping $I' = cI$ that also satisfies the same properties. Property (H4) is the most interesting one, it states that the integral is invariant with respect to the group operation.

The existence and uniqueness of the integral are addressed in the following theorem.

Theorem 2. An integral as defined above exists on a compact topological group G and is unique.

We'll approach Theorem 2 by proving small lemmas. Before that, let's however make a few definitions and fix some notation. From now on we'll fix a compact topological group G. If $A = (a_1, \dots, a_n)$ is a finite sequence of elements of G, we set

$$M(A, f; x) = \frac{1}{n} \sum_{k=1}^{n} f(xa_k).$$

Lemma 8. Let $A = (a_1, \dots, a_n)$ be a finite subsequence of elements of G and $f \in C(G)$. Then the mapping $M(A, f)$, $x \mapsto M(A, f; x)$, is continuous and satisfies the following inequalities:

$$\begin{aligned} \inf M(A, f) &\geq \inf f \\ \sup M(A, f) &\leq \sup f \\ \operatorname{osc} M(A, f) &\leq \operatorname{osc} f. \end{aligned}$$

Moreover if $B = \{b_1, \dots, b_m\}$ is also a finite subsequence of elements of G, then

$$M(A, M(B, f)) = M(AB, f),$$

where AB is the subsequence $(a_i b_j : 1 \leq i \leq n, 1 \leq j \leq n)$.

Proof. The continuity of $M(A, f)$ is clear. The first two inequalities are implied by

$$M(A, f; x) = \frac{1}{n} \sum_{k=1}^{n} f(xa_k) \geq \frac{1}{n} \sum_{k=1}^{n} \inf_{x \in G} f(x) = \inf_{x \in G} f(x),$$

and

$$M(A, f; x) = \frac{1}{n} \sum_{k=1}^{n} f(xa_k) \leq \frac{1}{n} \sum_{k=1}^{n} \sup_{x \in G} f(x) = \sup_{x \in G} f(x).$$

The third inequality follows from the first two since

$$\operatorname{osc} M(A, f) = \sup M(A, f) - \inf M(A, f) \leq \sup f - \inf f = \operatorname{osc} f.$$

Finally

$$\begin{aligned} M(A, M(B, f); x) &= \frac{1}{n} \sum_{k=1}^{n} M(B, f; xa_k) = \frac{1}{nm} \sum_{k=1}^{n} \sum_{j=1}^{m} f(xa_k b_j) \\ &= M(AB, f; x). \end{aligned}$$

□

Lemma 9. Suppose that $f \in C(G)$ is not constant. Then there exists a finite subsequence A of elements of G such that

$$\operatorname{osc} M(A, f) < \operatorname{osc} f.$$

Proof. Let $U = f^{-1}(-\infty, \sup f)$. Because f is continuous, U is open and the sets Ux^{-1}, $x \in G$, cover G. Because G is compact, there exists a finite subcover $Ua_1^{-1}, \dots, Ua_n^{-1}$ where $a_1, \dots, a_n \in G$. Now let $A = (a_1, \dots, a_n)$ and let $x \in G$

be arbitrary. Then x can be written in the form $x = ua_j^{-1}$ for some $u \in U$ and $1 \leq j \leq n$. Now we have

$$M(A, f; x) = \frac{1}{n}\sum_{k=1}^{n} f(xa_k) = \frac{1}{n}\sum_{k\neq j} f(xa_k) + \frac{f(xa_j)}{n}$$
$$< \frac{n-1}{n}\sup f + \frac{\sup f}{n} = \sup f.$$

Because G is compact, also $\sup M(A, f) < \sup f$ and hence

$$\operatorname{osc} M(a, f) = \sup M(A, f) - \inf M(A, f) < \sup f - \inf f = \operatorname{osc} f. \qquad \square$$

Definition 6. A real number p is a **right mean** of $f \in C(G)$ if for every $\varepsilon > 0$ there exists a finite subsequence $A = (a_1, \ldots, a_n)$ of elements of G such that

$$|M(A, f; x) - p| < \varepsilon$$

for all $x \in G$.

Lemma 10. Every continuous function $f \in C(G)$ admits a right mean.

Proof. Let

$$\Delta = \{M(A, f) : A \text{ is a finite subsequence of elements of } G\}.$$

We shall show that Δ is an equicontinuous family of continuous functions. Suppose that $\varepsilon > 0$ is given. Because f is continuous, it is also uniformly continuous by Lemma 2, and hence there exists a neighbourhood U of identity such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in G$ satisfying $xy^{-1} \in U$. Thus for any $x, y \in G$ and $M(A, f) \in \Delta$ we have

$$|M(A, f; x) - M(A, f; y)| = |\frac{1}{n}\sum_{k=1}^{n}(f(xa_k) - f(ya_k))|$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}|f(xa_k) - f(ya_k)| < \varepsilon,$$

whenever $xa_k a_k^{-1}y^{-1} = xy^{-1} \in U$. This means that the family Δ is equicontinuous. Moreover, Δ is uniformly bounded since f is bounded by some constant M and hence

$$|M(A, f; x)| \leq \frac{1}{n}\sum_{k=1}^{n}|f(xa_k)| \leq M.$$

Let now $s = \inf_{M(A,f)\in\Delta} \operatorname{osc} M(A, f)$. Then there exists a sequence of functions $f_k \in \Delta$ so that

$$\lim_{k\to\infty} \operatorname{osc} f_k = s.$$

By Theorem 1 there exists a uniformly convergent subsequence g_k of f_k. Let g be the limit of g_k so that $\operatorname{osc} g = s$. We'll show that g is a constant, which will then imply that $s = 0$. Suppose that g isn't constant. Then by Lemma 9 there exists a finite set A of elements of G so that

$$\operatorname{osc} M(A, g) = s' < s.$$

Because g_k converge uniformly to g, we get that there exists $k \in \mathbb{N}$ such that

$$|g_k(x) - g(x)| < \frac{s - s'}{3}$$

for all $x \in G$. Hence

$$|M(A, g_k; x) - M(A, g; x)| < \frac{s - s'}{3}$$

and thus

$$\operatorname{osc} M(A, g_k) \leq s' + \frac{2(s - s')}{3} < s.$$

Notice that $g_k = M(A', f)$ for some finite set A', so by Lemma 8 we have

$$M(A, g_k) = M(A, M(A', f)) = M(AA', f) \in \Delta.$$

This is a contradiction since s was chosen to be the infimum of oscillations of functions in Δ. Thus $g(x) = p$ is a constant.

This immediately implies that p is a right mean of f, since by the uniform convergence for any $\varepsilon > 0$ there exists $g_k \in \Delta$ such that

$$|g_k(x) - p| < \varepsilon$$

for all $x \in G$. □

Next we'll make analogous statements for the functions

$$M'(A, f; x) = \frac{1}{n} \sum_{k=1}^{n} f(a_k x).$$

We call a number p a **left mean** of f if for any $\varepsilon > 0$ there exists a finite sequence A of elements of G such that

$$|M'(A, f; x) - p| < \varepsilon$$

for all $x \in G$.

Lemma 11. Let A, B be finite sequences of elements of G. Then we have

$$M(A, M'(B, f)) = M'(B, M(A, f)).$$

Proof. By direct calculation

$$\begin{aligned} M(A, M'(B, f); x) &= \frac{1}{n} \sum_{j=1}^{n} M'(B, f; x a_j) \\ &= \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} f(b_k x a_j) \\ &= \frac{1}{m} \sum_{k=1}^{m} M(A, f; b_k x) = M'(B, M(A, f); x) \end{aligned}$$

for all $x \in G$. □

Lemma 12. Every continuous function $f \in C(G)$ admits a left mean.

Proof. Define a new topological group G' by setting $G' = G$ as a topological space and defining the group multiplication $\times\colon G' \times G' \to G'$ by $a \times b = ba$. We indeed have a group:

- Associativity: $a \times (b \times c) = a \times (cb) = cba = (ba) \times c = (a \times b) \times c$
- Identity: $a \times e = e \times a = a$
- Inverses are the same as in G: $a \times a^{-1} = a^{-1} \times a = e$

G' is also a topological group since $\times$ is given by $(a, b) \mapsto (b, a) \mapsto ba$ and is hence clearly continuous. By Lemma 10 there exists a right mean p for f in G'. Thus for any $\varepsilon > 0$ there exists a finite sequence A of elements of G such that

$$|M_{G'}(A, f; x) - p| < \varepsilon.$$

But

$$M_{G'}(A, f; x) = \frac{1}{n}\sum_{k=1}^{n} f(x \times a_k) = \frac{1}{n}\sum_{k=1}^{n} f(a_k x),$$

so $M_{G'} = M'_G$ and hence p is a left mean for f in G. □

Lemma 13. Let $f \in C(G)$ and suppose that A is a finite sequence of elements of G. Suppose that $|M(A, f; x) - p| < \varepsilon$ for all $x \in G$. Then

$$|M'(B, M(A, f); x) - p| < \varepsilon$$

for all finite sequences B of elements of G and $x \in G$.

Similarly if B is a finite sequence of elements of G and $|M'(B, f; x) - p| < \varepsilon$ for all $x \in G$, then

$$|M(A, M'(B, f); x) - p| < \varepsilon$$

for all finite sequences A of elements of G and $x \in G$.

Proof. Let $B = (b_1, \ldots, b_m)$. By direct calculation we have

$$\begin{aligned} |M'(B, M(A, f); x) - p| &= \left|\frac{1}{m}\sum_{j=1}^{m}(M(A, f; b_j x) - p)\right| \\ &\le \frac{1}{m}\sum_{j=1}^{m}|M(A, f; b_j x) - p| < \varepsilon. \end{aligned}$$

A similar calculation proves the other inequality. □

Lemma 14. For any $f \in C(G)$ there exist unique left and right means and they are the same.

Proof. It is enough to prove that if p is a right mean of f and q is a left mean of f, then $p = q$. Choose an arbitrary $\varepsilon > 0$. Then there exists a finite sequence $A = (a_1, \ldots, a_n)$ of elements of G such that

$$|M(A, f; x) - p| < \varepsilon.$$

for all $x \in G$. Similarly there exists a finite subsequence $B = (b_1, \ldots, b_m)$ of elements of G such that

$$|M'(B, f; x) - q| < \varepsilon.$$

for all $x \in G$. By Lemma 13

$$|M'(B, M(A, f); x) - p| < \varepsilon \text{ and } |M(A, M'(B, f); x) - q| < \varepsilon$$

for all $x \in G$. Together with Lemma 11 this implies that $|p - q| \leq 2\varepsilon$ and because ε was arbitrary, $p = q$. □

In view of the above lemma, we can make the following definition.

Definition 7. The **mean** $M(f)$ of a continuous function $f \in C(G)$ is either the right or the left mean of f.

Lemma 15. For any $f, g \in C(G)$ we have $M(f + g) = M(f) + M(g)$.

Proof. Let $p = M(f)$ and $q = M(g)$. We'll first show that p is a right mean of $M(B, f)$ for any finite sequence B of elements of G. Indeed because p is a left mean of f, for any $\varepsilon > 0$ there exists a finite sequence A of elements of G such that

$$|M'(A, f; x) - p| < \varepsilon.$$

Thus by Lemma 13 we have

$$|M(B, M'(A, f); x) - p| < \varepsilon,$$

and by Lemma 11 this is the same as

$$|M'(A, M(B, f); x) - p| < \varepsilon,$$

from which the claim follows.

Choose now an arbitrary $\varepsilon > 0$. Because q is a right mean of g, there exists a finite sequence B of elements of G so that

$$|M(B, g; x) - q| < \varepsilon$$

for all $x \in G$. Similarly because p is a right mean of $M(B, f)$, there exists a finite sequence A of elements of G so that

$$|M(A, M(B, f); x) - p| < \varepsilon,$$

which by Lemma 8 is the same as

$$|M(AB, f; x) - p| < \varepsilon.$$

Now if we write $A = (a_1, \ldots, a_n)$, then

$$|M(AB, g; x) - q| = |M(A, M(B, g; x)) - q| = |\frac{1}{n} \sum_{k=1}^{n} (M(B, g; xa_k) - q)\,|$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} |M(B, g; xa_k) - q| < \varepsilon$$

for all $x \in G$. Therefore

$$|M(AB, f + g; x) - (p + q)| \leq 2\varepsilon$$

holds for all $x \in G$, implying that $p + q$ is a right mean of $f + g$. □

We'll now prove Theorem 2.

Proof. Set $\int f(x)\,dx = M(f)$. The properties (H1),(H2),(H3) and (H5) are clearly satisfied. It remains to show (H4) and uniqueness. Let's start with (H4). Fix an element $a \in G$ and set $f'(x) = f(xa)$. Now for any $\varepsilon > 0$ there exists a finite sequence A of elements of G so that

$$|M(A, f; x) - M(f)| < \varepsilon$$

holds for all $x \in G$. In particular this implies that

$$|M(Aa^{-1}, f'; x) - M(f)| < \varepsilon$$

for all $x \in G$ since

$$M(Aa^{-1}, f'; x) = \frac{1}{n}\sum_{k=1}^{n} f'(xa_k a^{-1}) = \frac{1}{n}\sum_{k=1}^{n} f(xa_k) = M(A, f; x).$$

Hence $M(f)$ is also a right mean of f', i.e.

$$\int f(xa)\,dx = \int f(x)\,dx$$

and (H4) holds.

Suppose then that $\int^* : C(G) \to \mathbb{R}$ satisfies the properties (H1)–(H5). We'll show that $\int^* = \int$. Indeed let $f \in C(G)$ and choose $\varepsilon > 0$. Then there exists a finite sequence A of elements of G so that

$$|M(A, f; x) - M(f)| < \varepsilon$$

holds for all $x \in X$. Now in the lectures we have shown that we have the triangle inequality for integrals, so that

$$\begin{aligned}\left|\int^* f(x)\,dx - M(f)\right| &= \left|\int^* M(A, f; x)\,dx - M(f)\right| \\ &\leq \int^* |M(A, f; x) - M(f)|\,dx < \varepsilon,\end{aligned}$$

which (because ε was arbitrary) shows that $\int^* f(x)\,dx = M(f) = \int f(x)\,dx$. $\square$