1 Twisted products

Suppose G_1 and G_2 are topological groups and X is a space, such that G_1 acts on X from the left and G_2 acts on X from the right. We also assume that these two actions *commute* i.e.

$$g(xg') = (gx)g'$$

for all $g \in G_1, g' \in G_2$ and $x \in X$.
In this case we say that X is a $G_1 - G_1$ *bispace*.

Example 1.1. *Group G acts on itself by multiplication both from the left and on the right and these actions obviously commute since*

$$g(xh) = (gx)h$$

for all $g, x, h \in G$ (this is just a reformulation of associativity of the multiplication). Hence G is a $G - G$ bispace in a natural way.
More generally if H, K are subgroups of G then G is an $H - K$ bispace in an obvious way.

Remark 1.2. *Any G-space X can be considered as a $G - \{e\}$ bispace, where trivial group $\{e\}$ acts on X in the only possible way from the right. Likewise if X is a right G-space, it can be thought of as an $\{e\} - G$ bispace.*

Suppose $G_1, G_2, \ldots, G_{n+1}$ are topological groups, $n \geq 1$ and $G_i - G_{i+1}$ bispace X_i is given for every $i = 1, \ldots, n$. We define a (left) action of the product group $G_2 \times \ldots \times G_n$ on the product space $X_1 \times X_2 \ldots X_n$ defined by

$$(g_2, g_3, \ldots, g_n)(x_1, x_2, \ldots, x_n) = (x_1 g_2^{-1}, g_2 x_2 g_3^{-1}, \ldots, g_i x_i g_{i+1}^{-1}, \ldots, g_n x_n).$$

The orbit space of this action is denoted

$$X_1 \underset{G_2}{\times} X_2 \underset{G_3}{\times} X_3 \underset{G_4}{\times} \ldots \underset{G_{n-1}}{\times} X_{n-1} \underset{G_n}{\times} X_n$$

and called the *twisted product* of bispaces $X_1, \ldots, X_n$.
The class of the element $(x_1, x_2, \ldots, x_n)$ in the twisted product $X_1 \underset{G_2}{\times} X_2 \underset{G_3}{\times} X_3 \underset{G_4}{\times} \ldots \underset{G_{n-1}}{\times} X_{n-1} \underset{G_n}{\times} X_n$ will be denoted as

$$[x_1, x_2, \ldots, x_n].$$

Twisted product $X_1 \underset{G_2}{\times} X_2 \underset{G_{3}3}{X} \ldots \underset{G_{n+1}n}{X}$ has a canonical structure of $G_1 - G_{n+1}$-bispace. Precisely put we define action of G_1 on $X_1 \underset{G_2}{\times} X_2 \underset{G_3}{\times} X_3 \underset{G_4}{\times} \ldots \underset{G_{n-1}}{\times} X_{n-1} \underset{G_n}{\times} X_n$ by the formula

$$g[x_1, x_2, \ldots, x_n] = [gx_1, x_2, \ldots, x_n].$$

This action is well-defined, since X_1 is a $G_1 - G_2$ bispace, so

$$[gx_1, x_2, \ldots, x_n] = [g(x_1 g_2^{-1}), g_2 x_2 g_3^{-1}, \ldots, g_n x_n].$$

It is easy to check that it satisfies the algebraic properties of action. Finally the continuity of this action follows from the commutative diagram

$$\begin{array}{ccc}
G_1 \times (X_1 \times X_2 \dots X_n) & \xrightarrow{\Phi'} & X_1 \times X_2 \dots X_n \\
\Big\downarrow {\scriptstyle \mathrm{id} \times \pi} & & \Big\downarrow {\scriptstyle \pi} \\
G_1 \times (X_1 \underset{G_2}{\times} X_2 \underset{G_3}{\times} X_3 \underset{G_4}{\times} \dots \underset{G_{n-1}}{\times} X_{n-1} \underset{G_n}{\times} X_n) & \xrightarrow{\Phi} & X_1 \underset{G_2}{\times} X_2 \underset{G_3}{\times} X_3 \underset{G_4}{\times} \dots \underset{G_{n-1}}{\times} X_{n-1} \underset{G_n}{\times} X_n
\end{array}$$

in a usual manner. Here Φ is G-action defined above, Φ' is a mapping defined by

$$\Phi'(g, x_1, \dots, x_n) = (gx_1, x_2, \dots, x_n),$$

so obviously continuous. Since $\mathrm{id} \times \pi$ is open and surjective, it is quotient.

Similarly we define right action of G_{n+1} on $X_1 \underset{G_2}{\times} X_2 \underset{G_3}{\times} X_3 \underset{G_4}{\times} \dots \underset{G_{n-1}}{\times} X_{n-1} \underset{G_n}{\times} X_n$ by the formula

$$[x_1, \dots, x_n, x_{n+1}] \cdot g = [x_1, \dots, x_n, x_{n+1}g].$$

In the same way, as for the G_1-action, one checks that this action is well-defined and continuous right action of G_{n+1}. It is also easy to see that left action of G_1 and right action of G_{n+1} commute, so twisted product $X_1 \underset{G_2}{\times} X_2 \underset{G_3}{\times} X_3 \underset{G_4}{\times} \dots \underset{G_{n-1}}{\times} X_{n-1} \underset{G_n}{\times} X_n$ is a $G_1 - G_{n+1}$ bispace.

2 Induced G-space

Suppose G is a topological group, H closed subgroup of G and X is an H-space. We apply the construction of twisted products in the special case, where we consider G an $G - H$ bispace and X an $H - \{e\}$-bispace. Hence G acts on itself by multiplication on the left and H acts on G by multiplication from the right,

$$g \cdot h = gh^{-1}.$$

By the general construction above we obtain a twisted product $G \underset{H}{\times} X$, which is thus an orbit space of $G \times X$ with respect to the action of H defined by

$$h(g, x) = (gh^{-1}, hx).$$

Elements of $G \underset{H}{\times} X$ will be denoted as $[g, x]$. Notice that for any $h \in H$ we have

$$[g, hx] = [gh, x].$$

Since we consider G to be a left G-space, $G \underset{H}{\times} X$ is a G-space, with action of G defined by

$$g[g', x] = [gg', x].$$

There is a natural H-mapping $i\colon X \to G \underset{H}{\times} X$ defined by $i(x) = [e, x]$. It is obviously continuous and to check that it is H-equivariant we observe that

$$i(hx) = [e, hx] = [h, x] = h[e, x] = hi(x).$$

The pair $(G \underset{H}{\times} X, i)$ is called **induced G-space** of the H-space X. We often abuse notation and also call the G-space $G \underset{H}{\times} X$ an induced G-space of X.

Examples 2.1. *1.Suppose $G = S^1$ and $H = \{1, -1\} = \mathbb{Z}_2$. H acts on $X = [0, 1]$ by*

$$(-1) \cdot x = 1 - x.$$

The space $G \underset{H}{\times} X$ is homeomorphic to the Möbius band, which, thus, has a structure of G-space. The verification of this claim and the description of the S^1-action on the Möbius band defined by this description is left to the reader as an exercise.

2. Similarly if we take G and H as above and let H act on $X = S^1$ by

$$(-1) \cdot z = \bar{z} = z^{-1},$$

then the induced space $G \underset{H}{\times} X$ is Klein's bottle.

The mapping $i\colon X \to G \underset{H}{\times} X$ is always injective. To verify this suppose $x, y \in X$ and $[e, x] = [e, y]$. This means that there is an $h \in H$ such that $(e, x) = h(e, y) = (h^{-1}, hy)$. This implies that $h = e$, so $x = hy = y$.
Mapping i is, thus, an injective continuous mapping, but in general it need not to be imbedding.

Proposition 2.2. *Suppose X is Hausdorff G-space, where G is locally compact. Then $G \underset{H}{\times} X$ is Hausdorff. Moreover if H is compact, $G \underset{H}{\times} X$ is Palais proper G-space and mapping $i\colon X \to G \underset{H}{\times} X$ is a closed embedding.*

Proof. Action of H on G, $h \cdot g = gh^{-1}$ is Borel proper, since for any compact subspace K of G we have

$$H(K|K) = \{h \in H \mid hK \cap K \neq \emptyset\} = H \cap KK^{-1},$$

which is compact, since H is closed. Since G is locally compact, Theorem 2.8 in [1] implies that this action is also Palais proper. It is easy to check that if X is Palais proper H-space and Y is arbitrary H-space, then diagonal action $g(x, y) = (gx, gy)$ on $X \times Y$ is Palais proper. Hence action of H on $G \times H$ is Palais proper, so the induced space $G \underset{H}{\times} X$ is Hausdorff by Proposition 2.4. in [1].
Suppose H is compact. Then canonical projection $p\colon G \times X \to G \underset{H}{\times} X$ is closed, so its

restriction to closed subset $\{(e,x) \mid x \in X\}$ is closed. Also the embedding $x \mapsto (e,x)$ is a closed embedding. Hence their composition, which happens to be exactly i, is closed.
The second projection $G \times X \to G$ composed with canonical projection $G \to G/H$ defines a continuous mapping $f\colon G \times X \to G/H$. This mapping factors through the orbit space $G \underset{H}{\times} X$, since

$$f(gh^{-1}, hx) = gh^{-1}H = gH = f(g,x).$$

Thus we obtain a well-defined continuous mapping $\bar{f}\colon G \underset{H}{\times} X \to G/H$ defined by

$$\bar{f}[g,x] = gH.$$

This mapping is G-equivariant, because

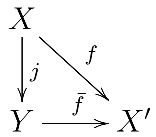
$$\bar{f}(g'[gx]) = \bar{f}[g'gx] = g'gH = g'(gH).$$

Now since H is compact, G/H is Borel proper (exercise 10.1). Since G (hence also G/H) is locally compact, Proposition 2.8 in [1] implies that G/H is Palais proper G-space. Since there exists a G-map $\bar{f}\colon G \underset{H}{\times} X \to G/H$, Proposition 2.8 in [1] implies that also $G \underset{H}{\times} X$ is Palais proper. □

Remark 2.3. *More generally it can be shown that $i\colon X \to G \underset{H}{\times} X$ is embedding also if canonical projection $\pi\colon G \to G/H$ has a local cross-section, i.e. there exists a neighbourhood U of eH in G/H and continuous mapping $s\colon U \to G$ such that $\pi \circ s = \mathrm{id}_U$ (see exercises). For example if G is a Lie group and H a closed subgroup of G, projection $\pi\colon G \to G/H$ always has a local cross-section. The proof of this extremely important property of Lie groups requires differential geometry, so we don't have a possibility to go through it.*

Induced G-space satisfies the important universal property defined below.

Definition 2.4. *Suppose X is an H-space, where H is a closed subgroup of a topological group G. The pair (Y,j), where Y is a G-space and $j\colon X \to Y$ an H-mapping is called* **universal G-space** *for X if for every G-space X' and H-mapping $f\colon X \to X'$ there exists unique G-mapping $\bar{f}\colon Y \to X'$ such that $f = \bar{f} \circ j$.*
In other words the diagram



commutes and $\bar{f}$ is the only mapping that makes it commute.

Universal G-space always exist and is unique up to a canonical G-homeomorphism.

Proposition 2.5. *Suppose X is an H-space as above. Then the pair $(G \underset{H}{\times} X, i)$ is a universal G-space for X. Moreover if (Y, j) is another universal G-space there exists unique G-homeomorphism $k\colon G \underset{H}{\times} X \to Y$ such that $k \circ i = j$. This homemorphism is defined by the formula*

$$k[g, x] = gj(x).$$

Proof. Suppose X' is an arbitrary G-space and $f\colon X \to X'$ is an H-mapping. Suppose $\bar{f}\colon G \underset{H}{\times} X \to X'$ is G-mapping such that $\bar{f} \circ i = f$. Then for all $g \in G, x \in X$ we have

$$\bar{f}([g, x]) = \bar{f}(g[e, x]) = g\bar{f}[e, x] = g\bar{f}(i(x)) = gf(x).$$

Hence $\bar{f}$ is unique. To show existence let us define $\bar{f}\colon G \underset{H}{\times} X \to X'$ by the formula above,

$$\bar{f}([g, x]) = gx.$$

We need to show that $\bar{f}$ is well-defined. Suppose $h \in H$. Then

$$\bar{f}([gh^{-1}, hx]) = (gh^{-1})hx = gx = \bar{f}([g, x]).$$

Hence $\bar{f}$ is well-defined. It is evidently continuous, since $\bar{f} \circ p = k$, where $k\colon G \times X \to X'$, $k(g, x) = gf(x)$ is clearly continuous and canonical projection $p\colon G \times X \to G \underset{H}{\times} X$ is a quotient mapping.
$\bar{f}$ is clearly G-mapping,

$$\bar{f}(g'[g, x]) = \bar{f}([g'g, x]) = (g'g)x = g'(gx) = g'f([g, x])$$

and $\bar{f}(i(x)) = \bar{f}([e, x]) = ex = x$ for all $x \in X$.
Hence $(G \underset{H}{\times} X, i)$ is a universal G-space for X.

Suppose (Y, j) is a universal G-space for X. Then in particular $j\colon X \to Y$ is H-mapping to a G-space X, so there exists unique $\bar{j}\colon G \underset{H}{\times} X \to Y$, such that $\bar{j} \circ i = j$. Likewise, since (Y, j) is a universal G-space for X and $i\colon X \to G \underset{H}{\times} X$ is an H-mapping, there exists unique $\bar{i}\colon Y \to G \underset{H}{\times} X$ such that $\bar{i} \circ j = i$.
Notice that by the explicit construction we have above $\bar{j}$ is defined by the formula

$$\bar{j}([g, x]) = gj(x).$$

Consider the mapping $\delta = \bar{i} \circ \bar{j}\colon G \underset{H}{\times} X \to G \underset{H}{\times} X$. This is a G-mapping and $\delta \circ i = i$, because

$$\delta \circ i = \bar{i} \circ \bar{j} \circ i = \bar{i} \circ j = i.$$

But on the other hand identity mapping $\mathrm{id}\colon G \underset{H}{\times} X \to G \underset{H}{\times} X$ is also a G-mapping that clearly has property $\mathrm{id} \circ i = i$. Since such a mapping is unique (by definition of universal G-mapping) we must have $\bar{i} \circ \bar{j} = \delta = \mathrm{id}$.

Similarly if we consider mapping $\sigma = \bar{j} \circ \bar{i} \circ Y \to Y$, we see that σ is a G-mapping such that $\sigma \circ j = j$. Since $\mathrm{id}\colon Y \to Y$ have the same properties and Y is a universal G-space, we have that $\bar{j} \circ \bar{i} = \sigma = \mathrm{id}$.

We have shown that $\bar{j}$ and $\bar{i}$ are inverses of each other. In particular they are both G-homeomorphisms. □

Induced space is a " transitive" construction.

Proposition 2.6. *Suppose G is a topological group, H, K subgroups of G and $H \subset K$. Suppose X is an H-space. Then there is a canonical G-homeomorphism $f\colon G \underset{K}{\times} (K \underset{H}{\times} X)) \to G \underset{H}{\times} X$ defined by*

$$f([g,[k,x]]) = [gk,x], g \in G, k \in K, x \in X.$$

Proof. Let $j\colon X \to G \underset{K}{\times} (K \underset{H}{\times} X))$ be defined by $j(x) = [e,[e,x]]$. Then j is continuous H-mapping. Let us show that the pair $(G \underset{K}{\times} (K \underset{H}{\times} X)), j)$ is a universal G-mapping for H-space X. Suppose $\alpha\colon X \to X'$ is an H-map, where X' is a G-space. Since X' is in particular K-space there exists unique K-mapping $\bar{\alpha}\colon K \underset{H}{\times} X \to X'$ such that $\bar{\alpha} \circ i = \alpha$, where $i\colon X \to K \underset{H}{\times} X$, $i(x) = [e,x]$. Moreover

$$\bar{\alpha}[k,x] = k\alpha(x)$$

for all $k \in K, x \in X$.
Now $\bar{\alpha}$ is a K-mapping into G-space X', so there exists unique mapping $\bar{\bar{\alpha}}\colon G \underset{K}{\times} (K \underset{H}{\times} X)) \to X'$ such that $\bar{\bar{\alpha}} \circ i' = \bar{\alpha}$, where $i'\colon K \underset{H}{\times} X \to G \underset{K}{\times} (K \underset{H}{\times} X))$ defined by $i'([k,x]) = [e,[k,x]]$. Moreover by construction we have

$$\bar{\bar{\alpha}}([g,[k,x]) = g\bar{\alpha}([k,x]) = gk\alpha(x).$$

Now simple calculation verifies that $\bar{\bar{\alpha}} \circ j = \alpha$, since $i' \circ i = j$.

We need to show uniqueness of $\bar{\bar{\alpha}}$. Suppose $\beta\colon G \underset{K}{\times} (K \underset{H}{\times} X)) \to X'$ is a G-mapping such that $\beta \circ j = \alpha$. Then

$$\beta([g,[k,x]]) = \beta(g[e,[k,x]]) = g\beta([e,[k,x]) = g\beta([e,k[e,x]]) =$$
$$= g\beta([k,[e,x]] = g\beta(k[e,[e,x]]) = gk\beta(j(x)) = gk\alpha(x).$$

Hence $\beta = \bar{\bar{\alpha}}$ is unique.

We have shown that the pair $(G \underset{K}{\times} (K \underset{H}{\times} X), j)$ is a universal G-mapping for H-space X. By the previous proposition there exists unique G-homeomorphism $f'\colon G \underset{H}{\times} X \to G \underset{K}{\times} (K \underset{H}{\times} X)$ and

$$f'([g,x]) = gj(x) = g[e,[e,x]] = [g,[e,x]].$$

It remains to show that f defined above is inverse to f'. We have

$$f(f'([g,x])) = f([g,[e,x]]) = [ge,x] = [g,x],$$

$$f'(f([g,[k,x]])) = f'([gk,x]) = [gk,[e,x]] = [g,k[e,x]] = [g,[k,x]].$$

□

Lemma 2.7. *Suppose H acts trivially on X. Then $G \underset{H}{\times} X$ is canonically G-homeomorphic to $G/H \times X$, with G-homeomorphism being $[g,x] \mapsto (gH, x)$.*
In particular if Z is a one-point space, then $G \underset{H}{\times} Z \cong G/H$.

Proof. (Easy) exercise. □

Proposition 2.8. *Suppose H, K are subgroups of a topological group G and $H \subset K$. Then the mapping $f\colon G \underset{K}{\times} K/H \to G/H$, $f([g,kH]) = gkH$ is a G-homeomorphism.*

Proof. Let $Z = \{z\}$ be one-point space (considered as H-space). Previous lemma and Proposition 2.6 imply that there are G-homeomorphisms

$$G \underset{K}{\times} K/H \cong G \underset{K}{\times} (K \underset{H}{\times} Z) \cong G \underset{H}{\times} Z \cong G/H.$$

Moreover if you trace the explicit G-homeomorphisms from the proofs above, you will get f as a concrete G-homeomorphism. □

The induced space has the same orbit space as the original space.

Proposition 2.9. *Suppose X is an H-space, where H is a closed subgroup of a topological group G. Then the canonical H-mapping $i\colon X \to G \underset{H}{\times} X$, $i(x) = [e,x]$ defines a homeomorphism $\tilde{i}\colon X/H \cong (G \underset{H}{\times} X)/G$ between orbit spaces.*

Proof. Consider the composite mapping $p \circ i\colon X \to (G \underset{H}{\times} X)/G$, where $p\colon G \underset{H}{\times} X \to (G \underset{H}{\times} X)/G$ is a canonical projection. It is clearly continuous. Also $p \circ i$ factors through the orbit space X/H, since for all $x \in X, h \in H$ we have

$$p(i(hx)) = G[e,hx] = G[h,x] = Gh[e,x] = G[e,x] = p(i(x)).$$

Hence there is induced continuous mapping $\tilde{i}\colon X/H \cong (G \underset{H}{\times} X)/G$.

The composition of the second projection $G \times X \to X$ and canonical projection $X \to X/H$ is a continuous mapping $j\colon G \times X \to X/H$, $(g,x) \mapsto Hx$. Suppose $h \in H$. Then

$$j(gh^{-1}, hx) = Hhx = Hx,$$

so j factors through $G \underset{H}{\times} X$, hence defines a continuous mapping $\bar{j}\colon G \underset{H}{\times} X \to X/H$. Finally for any $g' \in G$

$$\bar{j}(g'[g,x]) = \bar{j}([g'g,x]) = Hx = \bar{j}([g,x]),$$

so $\bar{j}$ factors through the orbit space $(G \underset{H}{\times} X)/G$ and hence defines a continuous mapping $\tilde{j}\colon (G \underset{H}{\times} X)/G \to X/H$ defined by $\tilde{j}(G[g,x]) = Hx$.
Now

$$\tilde{j} \circ \tilde{i}(Hx) = \tilde{j}(G[e,x]) = Hx,$$
$$\tilde{i} \circ \tilde{j}(G[g,x]) = \tilde{i}(Hx) = G[e,x] = G[g,x]$$

for all $g \in G, x \in X$. Hence $\tilde{j}$ is a continuous inverse of $\tilde{i}$, so the latter is a homeomorphism. □

Proposition 2.10. *Suppose K is a closed subgroup of a topological group G and H is a normal closed subgroup of K. Let X be a K-space, such that the restricted action of H on X is trivial. Then the mapping $f\colon G \underset{K}{\times} X \to (G/H) \underset{K/H}{\times} X$ defined by $f([g,x]) = [gH, x]$ is a G-homeomorphism.*
Here we consider G/H a $G - K/H$-bispace.

Proof. Notice that since H acts trivially on X, there is a well-defined canonical action of K/H on X defined by $kH \cdot x = kx$, hence X can be considered anK/H-space.
The fact that G/H is a left G-space is well-known. Let us check that also the right action of K/H on G/H defined by $(gH) \cdot (kH) = gkH$ is well-defined. Suppose $h, h' \in H$. Then

$$(gh)(kh')H = g(hkh')H = ghkH = gkh''H = gkH$$

for some $h'' \in H$, since H is normal in K. Continuity of this action is checked in the usual manner and the fact that left G-action and right K/H-action commute is easy to verify.

Continuity of f follows as usual - it is a mapping induced by the mapping $(g,x) \mapsto [gH, x]$ which is clearly continuous, one only needs to verify that f is well-defined. Suppose $k \in K$. Then

$$f([gk^{-1}, kx]) = [gk^{-1}H, kx] = [gH \cdot (kH)^{-1}, (kH)x] = [gH, x] = f([g,x]).$$

To define an inverse mapping $f'\colon (G/H) \underset{K/H}{\times} X \to G \underset{K}{\times} X$ we first consider canonical projection $p\colon G \times X \to G \underset{K}{\times} X$ and show that it induced well-defined continuous mapping $p'\colon G/H \times X \to G \underset{K}{\times} X$. Suppose $g' = gh$ for some $h \in H$. Then

$$p(g', x) = [g', x] = [gh, x] = [g, hx] = [g, x],$$

since $H \subset K$ and H acts trivially on X. Hence p' exists. Since $\pi \times \mathrm{id}\colon G \times X \to G/H \times X$ is a quotient mapping (it is a product of open surjections) and $p' \circ (\pi \times \mathrm{id}) = p$ is continuous, it follows that p' is continuous.

Next we show that p' factors through the quotient space $(G/H) \underset{K/H}{\times} X$. Suppose $kH \in K/H$. Then

$$p'((gH{\cdot}kH^{-1}, (kH)x)) = p'(gk^{-1}H, kx) = p(gk^{-1}, kx) = [gk^{-1}, kx] = [g, x] = p'(gH, x).$$

Hence there is a well-defined continuous mapping $f' \colon (G/H) \underset{K/H}{\times} X \to G \underset{K}{\times} X$, defined by the formula

$$f'[gH, x] = [g, x].$$

f and f' are clearly inverses of each other. Also for instance f is G-mapping:

$$f(g[g', x]) = f([gg', x]) = [gg'H, x] = g[g'H, x].$$

□

Corollary 2.11. *Suppose H is a closed subgroup of a topological space G and X is a G-space. Then*

$$G \underset{N(H)}{\times} X^H \cong G/H \underset{N(H)/H}{\times} X^H.$$

as G-spaces.

Proof. X^H is an $N(H)$-space, and the restricted action of H on X^H is trivial, so the previous proposition applies. □

Consider the inclusion $\iota \colon X^H \hookrightarrow X$, where X is a G-space and H is a closed subgroup of G. Then ι is $N(H)$-equivariant, so by the universal property of induced G-space there is continuous G-mapping $f \colon G \underset{N(H)}{\times} X^H \to X$, defined by $f([g, x]) = gx$ for all $g \in G, x \in X$. Since by the previous corollary there is a G-homeomorphism

$$G \underset{N(H)}{\times} X^H \cong G/H \underset{N(H)/H}{\times} X^H,$$

so there is also a G-mapping $f' \colon G/H \underset{N(H)/H}{\times} X^H \to X$ defined by $f'([gH, x]) = gx$.

Theorem 2.12. *Suppose G is a compact group, H its closed subgroup and X a G-space, such that only one isotropy type $[H]$ occurs at X. Then both $f \colon G \underset{N(H)}{\times} X^H \to X$ and $f' \colon G/H \underset{N(H)/H}{\times} X^H \to X$ are G-homeomorphisms.*

Proof. Enough to prove that f is a homeomorphism. We first show that f is surjective. Suppose $x \in X$. Then $[G_x] = [H]$, so there exists $g \in G$ such that $G_x = gHg^{-1}$. It follows that

$$G_{g^{-1}x} = g^{-1}G_x g = H,$$

so $y = g^{-1}x \in X^H$ and

$$f([g, y]) = gy = x.$$

Thus f is a surjection.
Next we prove f is an injection.
Suppose $[g,x],[g',x'] \in G \underset{N(H)}{\times} X^H$ such that $f([g,x]) = gx = g'x' = f([g',x'])$. Let $n = g'^{-1}g$, then $nx = x'$, so

$$H \subset G_{x'} = nG_x n^{-1},$$

since $x' \in X^H$. By assumptions G_x is conjugate to H. Since G is compact, by Lemma 1.15 in [4] this implies that in $n \in N(H)$. Hence

$$[g,x] = [g'n,x] = [g',nx] = [g',x'].$$

We have shown that f is bijection.

Since G is compact, the action mapping $\Phi\colon G \times X \to X$ is closed (Theorem 1.9 in [2]). Since X^H is closed, the restricted mapping $\phi = \Phi|\colon G \times X^H \to X$ is closed. Now by definition of f we have that $f \circ \pi = \phi$, so ϕ is surjective (since f is). As a surjective closed mapping, it is a quotient mapping. Here $\pi\colon G \times X^H \to G \underset{N(H)}{\times} X^H$ is a canonical projection, hence quotient. It follows that the induced mapping f is a homeomorphism. □

The meaning of this theorem is that, under assumptions we made, the restricted action of $N(H)$ on the subspace X^H contains all the information about the space.

Proposition 2.13. *Suppose X, G and H are as in Theorem 2.12 above. Then inclusion $X^H \hookrightarrow X$ induces homeomorphism*

$$X^H/N(H) \cong X/G$$

between orbit spaces.

Proof. By Proposition 2.9 inclusion $X^H \hookrightarrow X$ induces a homeomorphism $X^H/H \cong (G \underset{N(H)}{\times} X^H)/G$. On the other hand by the previous Theorem $G \underset{N(H)}{\times} X^H \cong X$ as G-spaces, so also orbit spaces are homeomorphic, $(G \underset{N(H)}{\times} X^H)/G \cong X/G$. Combining these two result together yields the claim. □

References

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