Fibre bundles

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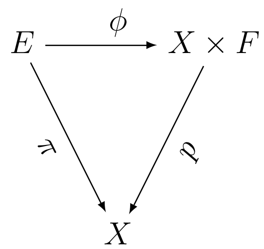
1 Fibre bundles and bundle maps

Let us begin with a simple example.

Example 1. *Let E and X be topological spaces and $\pi : E \to X$ a continuous mapping between them. Suppose that for some topological space F there exists a homeomorphism $\phi : E \to X \times F$ such that $p \circ \phi = \pi$, where p is the first projection, meaning that the diagram*



commutes. Then the triple (E, π, X) is called a product bundle, where E is the total space, X is the base space and π is the projection.

In the example we just took the cartesian product of X and F and wished for a mapping such that the diagram would commute. A fibre bundle can be seen as a generalization of this in the sense that we define it to look like a product bundle but only locally. In almost any non-trivial case, the total space of a fibre bundle may be quite difficult to handle as such but, nevertheless, locally it behaves as a cartesian product. Let us now give the formal definition, which contains a bit more than the previous discussion implies.

Definition 1. *Let K be a topological group acting effectively on a topological space F, and let E and X be Hausdorff spaces with a continuous mapping $\pi : E \to X$. Suppose that there exists an open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of X and for each $\alpha \in \Lambda$ a homeomorphism $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ with the following properties:*

1. *The first projection* $p_\alpha : U_\alpha \times F \to U_\alpha$ *gives* $p_\alpha \circ \phi_\alpha = \pi|_{\pi^{-1}U_\alpha}$

2. *For* $x \in U_\alpha$*, first define*

$$\phi_{\alpha,x} : \pi^{-1}(x) \to F$$

   by $\phi_{\alpha,x}(z) = (p'_\alpha \circ \phi_\alpha)(z)$*, where* $p'_\alpha : U_\alpha \times F \to F$ *is the second projection. Put*

$$\theta_{\beta\alpha}(x) = \phi_{\beta,x} \circ \phi_{\alpha,x}^{-1} : F \to F \qquad \text{for } x \in U_\alpha \cap U_\beta.$$

   Then $\theta_{\beta\alpha}(x) : F \to F$ *belongs to* K *for each* x*.*

3. *The mapping* $\theta_{\beta\alpha} : U_\alpha \cap U_\beta \to K$ *is continuous.*

The system $(E, \pi, X, F, K, U_\alpha, \phi_\alpha)$ *so obtained is called a* coordinate bundle.

Two coordinate bundles $(E, \pi, X, F, K, U_\alpha, \phi_\alpha)$ and $(E, \pi, X, F, K, \phi'_\mu, U'_\mu)$ are considered to be equivalent, if for each x the values $\bar{\theta}_{\mu\alpha}(x) = \phi'_{\mu,x} \circ \phi_{\alpha,x}^{-1}$ belong to K and the mapping $\bar{\theta} : U'_\mu \cap U_\alpha \to K$ is continuous. An equivalence class of a coordinate bundle is called a *fibre bundle*, with E the total space, X the base space, π the projection, F the fibre and K the structure group. For a representative, or a coordinate bundle, U_α is called a coordinate neighbourhood, ϕ_α a coordinate mapping and $\theta_{\beta\alpha}$ a transition function.

If x is in the base space, many points of the total space E may project onto it: namely the points $z \in \pi^{-1}(x)$. Each of these has different representations with respect to different charts U_α containing x. We are interested to change the chart, i.e. to deduce the new coordinates of a point when moving between overlapping charts U_α and U_β. This is exactly what the mapping $\theta_{\beta\alpha}(x)$ does: if $x \in U_\alpha \cap U_\beta$ and $z \in \pi^{-1}(x)$, then $\phi_\alpha(z) = (x, f)$ contains the necessary information in the second coordinate; hence we define $\phi_{\alpha,x}(z) = p_2 \circ \phi_\alpha(z)$. Now the mapping $\theta_{\beta\alpha}(x) = \phi_{\beta,x} \circ \phi_{\alpha,x}^{-1}$ gives us the second coordinate of the point z w.r.t. U_β when we know it w.r.t. U_α.

Furthermore, we demand that $\theta_{\beta\alpha}$ takes the points from the overlapping charts to elements of K in a continuous manner so that we get a grasp of the structure of this transition process.

According to the above discussion we can write

$$\phi_\beta \circ \phi_\alpha^{-1}(x, f) = \phi_\beta(\phi_{\alpha,x}^{-1}(f)) = (x, \phi_{\beta,x} \circ \phi_{\alpha,x}^{-1}(f)) = (x, \theta_{\beta\alpha}(x)(f)),$$

which is often used to define the mapping $\theta_{\beta\alpha}$.

Example 2. *The product bundle defined in the beginning actually fulfills the demands of the above definition.*

Example 3. *Consider for a moment the famous Möbius band. It is a great example of such a structure that locally looks like a cartesian product of two spaces but which doesn't hold globally because of the twist. In the following we will show that the Möbius band turns into a coordinate bundle in a natural way.*

Let $I = [-1, 1]$ and choose I for the fibre F and a circle obtained from I for the base space X. The total space E, the actual Möbius strip, is then decided to be the product space $I \times F$ with a twisting identification $(1, y) \sim (-1, 1-y)$; formally E is the quotient space $I \times F/\sim$. Note that this twist is done over the central line $\{(x, 0) : x \in I\}$ of $I \times F$ meaning that it only identifies the two ends making it coincide with the base space X. Now we get a projection $\pi : E \to X$ by taking just the first projection $I \times F \to I$ w.r.t. the identification. Formally $\pi([x, y]_\sim) = [x, 0]_\sim$ where $[,]_\sim$ denotes the equivalence class. So π just projects a point from the strip straight to the central circle.

Now it suffices to choose an open cover for X together with the coordinate functions to make the above construction a coordinate bundle. Using the identification $\sim$ of -1 and 1, define

$$U_\alpha = \{(-2/3, 2/3)\} / \sim = (-2/3, 2/3) \quad \text{and} \quad U_\beta = \{[-1, -1/3] \cup [1/3, 1]\} / \sim .$$

Then $\{U_\alpha, U_\beta\}$ is an open covering for the base space X, with

$$U_\alpha \cap U_\beta = (-2/3, -1/3) \cup (1/3, 2/3).$$

Continue to define mappings

$$\phi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times F, \qquad \phi_\alpha = \mathrm{Id}$$

and

$$\phi_\beta : \pi^{-1}U_\beta \to U_\beta \times F,$$
$$\phi_\beta([x, y]_\sim) = \begin{cases} (x, y), & x \in (1/3, 1], y \in F \\ (x, -y), & x \in [-1, -1/3), y \in F \end{cases}$$

(Note that ϕ_β is well-defined since we identified $(1, y)$ with $(-1, -y)$.)

Both mappings are clearly homeomorphisms satisfying $p \circ \phi_\alpha = \pi|_{\pi^{-1}U_\alpha}$ and $p \circ \phi_\beta = \pi|_{\pi^{-1}U_\beta}$, whence they are coordinate functions. Remembering that $\theta_{\beta\alpha}(x)$ only tells what happens with the F-coordinate when changing charts, we get

$$\theta_{\beta\alpha}(x) = \begin{cases} -\mathrm{Id}_F, & x \in (-2/3, -1/3) \\ \mathrm{Id}_F, & x \in (1/3, 2/3). \end{cases}$$

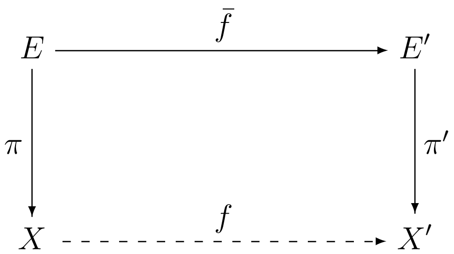
Now choose $K = \mathbb{Z}_2$ to be the cyclic group of two elements and let it act effectively on F by $g.y = -y$, where g is the generator of K. Under this action the elements of K correspond exactly to the mappings Id_F and $-\mathrm{Id}_F$, whence $\theta_{\beta\alpha}(x) \in K$ for any $x \in U_\alpha \cap U_\beta$. Moreover, $\theta_{\beta\alpha}$ is constant on the two components of the intersection and is thus continuous. Hence $\pi : E \to X$ is a coordinate bundle.

Definition 2. *Let $\xi = (E, \pi, X, F, K)$ and $\xi' = (E', \pi', X', F, K)$ be two fibre bundles with the fibre and structure group in common. A continuous mapping*

$$\bar{f} : E \to E'$$

is called a bundle map *from ξ to ξ' if the following two conditions hold.*

1. *There exists a continuous mapping $f : X \to X'$ between the base spaces such that $\pi' \circ \bar{f} = f \circ \pi$, giving us the following commutative diagram:*



2. *For any coordinate neighbourhoods U_α of ξ and V'_μ of ξ' such that $U_\alpha \cap f^{-1}V'_\mu \neq \emptyset$ we obtain a continuous mapping $f_{\mu\alpha} : U_\alpha \cap f^{-1}V'_\mu \to K$ when defining*

$$f_{\mu\alpha}(x) = \phi'_{\mu,f(x)} \circ \bar{f} \circ \phi^{-1}_{\alpha,x} \qquad \textit{for} \quad x \in U_\alpha \cap f^{-1}V'_\mu$$

If this happens while $X = X'$ and $f = \mathrm{Id}$, then the bundle map $\bar{f}$ is called an isomorphism *and the fibre bundles are said to be* isomorphic. *This is denoted $\xi \simeq \xi'$ or simply $E \simeq E'$. A fibre bundle isomorphic to a product bundle is called a* trivial bundle *or just* trivial.

Clearly $f_{\mu\alpha}(x)$ is a continuous mapping $F \to F$ for every $x \in X$. The crucial thing is to demand that the values of $f_{\mu\alpha}$ actually belong to K. The point of the above definition is that we want to find a continuous $f : X \to X'$ that makes the diagram commutative, and together with $\bar{f}$ gives transition functions for coordinate charts U_α and V'_μ overlapping in the sense $U_\alpha \cap f^{-1}V'_\mu \neq \emptyset$.

Remark. *If f happens to be a homeomorphism, then $\bar{f}$ is a homeomorphism and its inverse $\bar{f}^{-1}$ is a bundle map too.*

Proposition 1. *The projection $\pi : E \to X$ of a fibre bundle is an open mapping.*

Proof. Exercise. □

2 Principal bundles and associated bundles

Definition 3. *A fibre bundle is called a* principal fibre bundle *if $F = K$ and K acts on F by left translations, $k.f = kf$. For a principal bundle the total space is often denoted by P and the bundle by (P, π, X, K).*

Proposition 2. *For a principal K-bundle there is a canonical free right K-action on P such that each fibre $\pi^{-1}(x)$ is K-invariant w.r.t. this action. The projection π induces a homeomorphism $\bar{\pi} : P/K \to X$, where P/K denotes the orbit space of this new action.*

Proof. First define a right K-action on $U_\alpha \times K$ by

$$(x, k).k' = (x, kk') \qquad \text{for} \quad x \in U_\alpha,\ k, k' \in K.$$

This induces a right K-action on $\pi^{-1}U_\alpha$ via $\phi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times K$ when putting

$$z.k' = \phi_\alpha^{-1}(\phi_\alpha(z).k') \qquad \text{for} \quad z \in \pi^{-1}U_\alpha,\ k' \in K.$$

Let us show that the above action is independent from the choice of the coordinate function ϕ_α. For this, suppose $U_\alpha \cap U_\beta \neq \emptyset$ and $\phi_\beta : \pi^{-1}U_\beta \to U_\beta \times K$ is another coordinate function. For $z \in \pi^{-1}(U_\alpha \cap U_\beta)$ we have a unique $k \in K$ with $\phi_\alpha(z) = (\pi(z), k)$. Now

$$\begin{aligned} \phi_\beta \circ \phi_\alpha^{-1}(\phi_\alpha(z).k') &= \phi_\beta \circ \phi_\alpha^{-1}((\pi(z), k).k') = \phi_\beta \circ \phi_\alpha^{-1}(\pi(z), kk') \\ &= (\pi(z), \theta_{\beta\alpha}(\pi(z))(kk')) = (\pi(z), (\theta_{\beta\alpha}(\pi(z))k)k') \\ &= (\pi(z), (\theta_{\beta\alpha}(\pi(z))k)).k' = (\phi_\beta \circ \phi_\alpha^{-1}(\pi(z), k)).k' \\ &= \phi_\beta(z).k', \end{aligned}$$

whence $\phi_\alpha^{-1}(\phi_\alpha(z).k') = \phi_\beta^{-1}(\phi_\beta(z).k')$. This proves the claim and hence the induced action is well-defined. (In the equations we first used just the fact $\phi(z) = (\pi(z), k)$, then the definition of the action, then the definition of $\theta_{\beta\alpha}$, then associativity of actions, then the definition of the action, then the definition of $\theta_{\beta\alpha}$ and finally again the fact $\phi_\alpha(z) = (\pi(z), k)$.)

Now we've defined a right K-action on the total space P by defining it on the sets $\pi^{-1}U_\alpha$ that form an open covering of P. The action first defined is free because $(x,k).k' = (x,k).k''$ iff $kk' = kk''$ iff $k' = k''$. Since the second action is obtained through a homeomorphism from the first, it is also free. Moreover, since the actions don't affect the first coordinates, each fibre remains K-invariant under this action.

The projection $\pi : P \to X$ induces a continuous bijection $\bar{\pi} : P/K \to X$ where P/K is the orbit space of the free right action above. Since π is open, $\bar{\pi}$ is open too and hence a homeomorphism. □

Next we move to construct a so called associated bundle for a principal bundle. First let X be a right G-space and Y a left G-space. Then G operates on the space $X \times Y$ from the left by $g.(x,y) = (xg^{-1}, gy)$. Denote the orbit space $(X \times Y)/G = X \underset{G}{\times} Y$ - this is called the *twisted product* of the G-spaces X and Y. In the following we refer to the canonical right action from the last proposition as "the right K-action".

Proposition 3. *Let $\pi : P \to X$ be a principal K-bundle and suppose K acts effectively from the left on a topological space F. Then*

$$p : P \underset{K}{\times} F \to X,$$

where p is defined by $p([z,y]) = \pi(z)$ for $z \in P, y \in F$, is a fibre bundle with fibre F and structure group K.

Proof. Let $\phi : \pi^{-1}U \to U \times K$ be a coordinate function. Note that if $[z,y] = [z',y']$, then for some $k \in K$ we have $(z.k^{-1}, k.y) = (z',y')$ and hence $z.k^{-1} = z'$. Since the fibers of the principal bundle are invariant under the right K-action, we see that $\pi(z) = \pi(z')$, whence p is well-defined.

Corresponding to ϕ, let us construct a mapping $\bar{\phi} : p^{-1}U \to U \times F$ by composition of 4 mappings ϕ_1, ϕ_2, ϕ_3 and ϕ_4, defined as follows:

Since

$$p^{-1}U = \left\{ [z,y] \in P \underset{K}{\times} F : \pi(z) \in U \right\} = \left\{ [z,y] : z \in \pi^{-1}U \right\} = \pi^{-1}U \underset{K}{\times} F$$

we may let

$$\phi_1 = \mathrm{Id} : p^{-1}U \to \pi^{-1}U \underset{K}{\times} F.$$

The second map is yielded from ϕ by putting

$$\phi_2 = \phi \underset{K}{\times} \mathrm{Id} : \pi^{-1}U \underset{K}{\times} F \to (U \times K) \underset{K}{\times} F.$$

The third map comes from the fact that twisted product obeys associativity on the level of homeomorphy, so that

$$\phi_3 : (U \times K) \underset{K}{\times} F \equiv U \times (K \underset{K}{\times} F).$$

The fourth map is just the product of Id and the homeomorphism that collapses $K \underset{K}{\times} F \equiv F$:

$$\phi_4 : U \times (K \underset{K}{\times} F) \to U \times F, \qquad (u, [k, f]) \mapsto (u, k.f)$$

Because the mappings $\phi_1, \phi_2, \phi_3, \phi_4$ are all homeomorphisms, so is their composition $\bar{\phi} = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1 : p^{-1}U \to U \times F$.

Let us show that $\bar{\phi}$ is a coordinate function. For this, suppose $\psi : \pi^{-1}V \to V \times K$ is another coordinate function and let $\theta : U \cap V \to K$ be the transition function for ϕ and ψ, i.e. $\theta(x) = \psi_x \circ \phi_x^{-1}$. Let $\psi_1, \ldots, \psi_4$ be mappings defined as the mappings $\phi_1, \ldots, \phi_4$ and denote their composition by $\bar{\psi} : p^{-1}V \to V \times F$. For $x \in U \cap V$ identify F with $\{x\} \times F$. Then for any $y \in F$ we get

$$\begin{aligned}
\bar{\psi}_x \circ \bar{\phi}_x^{-1}(y) &= \bar{\psi}_x \circ \bar{\phi}^{-1}(x, y) = \bar{\psi}_x \circ \phi_1^{-1} \circ \phi_2^{-1} \circ \phi_3^{-1} \circ \phi_4^{-1}(x, y) \\
&= \bar{\psi}_x \circ \phi_2^{-1} \circ \phi_3^{-1}(x, [e, y]) = \bar{\psi}_x \circ \phi_2^{-1}([(x, e), y]) \\
&= \bar{\psi}_x([\phi_x^{-1}(e), y]) = \psi_4 \circ \psi_3 \circ \psi_2([\phi_x^{-1}(e), y]) \\
&= \psi_4 \circ \psi_3([(x, \psi_x \circ \phi_x^{-1}(e)), y]) \\
&= \psi_4 \circ \psi_3([(x, \theta(x)e), y]) = \psi_4(x, [\theta(x)e, y]) \\
&= (x, \theta(x)y) = \theta(x)y
\end{aligned}$$

meaning that $\bar{\psi}_x \circ \bar{\phi}_x^{-1} = \theta(x)$. Hence θ is the transition function for $\bar{\phi}$ and $\bar{\psi}$. □

Definition 4. *The bundle obtained in the last proposition is called the* bundle associated with the principal bundle P*.*

The idea is to turn a principal bundle into a fibre bundle with a new fibre but the same structure group. On the other hand, any fibre bundle carries a principal bundle with it, called the principal bundle associated with the fibre bundle. It is not hard to show the existence of this but we will not do it here. However, one can prove that the associated principal bundle defines the structure of the bundle in the following sense:

Proposition 4. *Two fibre bundles having the same base, fibre and structure group are isomorphic if and only if their associated principal bundles are isomorphic.*

Also, the principal bundles may be easier to handle than their associated bundles, and the previous statement is useful when deciding whether two bundles are isomorphic or not.

3 Fibre products and induced bundles

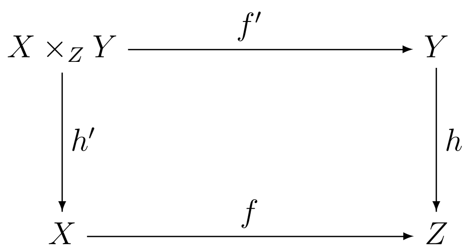
Definition 5. *If X, Y and Z are G-spaces with $f : X \to Z$, $h : Y \to Z$ G-maps, then the subspace*

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = h(y)\}$$

is called the fibre product *of X and Y.*

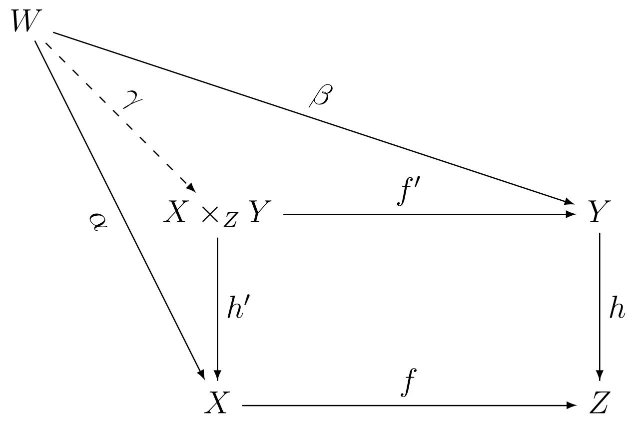
The fibre product is also called the pull-back of Y via f. It is a G-space under the diagonal G-action taking $(g, (x, y)) \mapsto (gx, gy)$: if $f(x) = h(y)$, then $f(gx) = g.f(x) = g.h(x) = h(gx)$, since f and h are assumed to be G-maps.

Let us now define mappings $f' : X \times_Z Y \to Z$, $h' : X \times_Z Y \to Z$ by just restricting the projections respectively. These are G-maps since the projections are G-maps. This leads the following diagram, called the *pull-back diagram*, to be commutative:



Fibre product has the following universal property:

Proposition 5. *If $X \times_Z Y$ is a fibre product and W is such a G-space that there exists G-maps $\alpha : W \to X$ and $\beta : W \to Y$ with $f \circ \alpha = h \circ \beta$, then there is a unique G-map $\gamma : W \to X \times_Z Y$ so that the following diagram commutes:*



Proof. Exercise. □

Next we should prove some basic facts of the pull-back construction.

Proposition 6. *1. If f is surjective or injective, then so is f'. If f is bijective, then so is f'.*

2. If f is open, then so is f'.
Similar results hold, of course, for h and h' too.

Proof. Exercise. □

The fibre product can be equipped with a structure of a fibre bundle in a natural way.

Proposition 7. *Let $\xi = (E, \pi, X, F, K)$ be a fibre bundle and $f : Y \to X$ a continuous function. Then the pull-back*

$$\begin{array}{ccc} Y \times_X E & \xrightarrow{\quad f' \quad} & E \\ \Big\downarrow \pi' & & \pi \Big\downarrow \\ Y & \xrightarrow{\quad f \quad} & X \end{array}$$

yields a fibre bundle $\pi' : Y \times_X E \to Y$ with fibre F and structure group K. Moreover, f' is a bundle map.

Proof. Since f is continuous, the sets $f^{-1}U_\alpha$, where U_α is a coordinate chart for ξ, form an open covering for Y. Define ψ_α by

$$\psi_\alpha : \pi'^{-1} f^{-1} U_\alpha \to f^{-1} U_\alpha \times F, \qquad \psi_\alpha(y, z) = (y, p'_\alpha \circ \phi_\alpha(z)),$$

where $p'_\alpha : U_\alpha \times F \to F$ is the projection. The mappings ψ_α are well-defined, since for any $(y, z) \in \pi'^{-1} f^{-1} U_\alpha$ we have $f(y) = \pi(z)$ and hence $z \in \pi^{-1}(f(y)) \subset \pi^{-1} U_\alpha$. We need to show that ψ_α is a homeomorphism. It has an inverse

$$\psi'_\alpha : f^{-1} U_\alpha \times F \to \pi'^{-1} f^{-1} U_\alpha$$

defined by

$$\psi'_\alpha(y, b) = (y, \phi^{-1}_{\alpha, f(y)}(b)),$$

because

$$\psi'_\alpha(\psi_\alpha(y, z)) = \psi'_\alpha(y, p'_\alpha \circ \phi_\alpha(z)) = (y, \phi^{-1}_{\alpha, f(y)}(p'_\alpha \circ \phi_\alpha(z))) = (y, z)$$

and similarly $\psi_\alpha(\psi'_\alpha(y,b)) = (y,b)$. One can easily check that ψ'_α is well-defined. The mappings ψ_α and ψ'_α are clearly continuous, whence ψ is a homeomorphism.

Note that for $y \in Y$ we have $\pi'^{-1}(y) = \{y\} \times \pi^{-1}(f(y))$ and define

$$\psi_{\alpha,y} : \pi'^{-1}(y) \to F \qquad \text{by} \qquad \psi_{\alpha,y} = q'_\alpha \circ \psi_\alpha$$

where q'_α denotes the second projection $f^{-1}U_\alpha \times F \to F$. But since $\psi_\alpha(y,z) = (y, p'_\alpha \circ \phi_\alpha(z))$, we get simply that $\psi_{\alpha,y}(y,z) = p'_\alpha \circ \phi_\alpha(z) = \phi_{\alpha,f(y)}(z)$. Hence, for $y \in f^{-1}U_\alpha \cap f^{-1}U_\beta$, we have

$$\psi_{\beta,y} \circ \psi^{-1}_{\alpha,y} = \phi_{\beta,f(y)} \circ \phi^{-1}_{\alpha,f(y)} = \theta_{\beta\alpha}(f(y)).$$

Denote $\tau_{\beta\alpha}(y) = \psi_{\beta,y} \circ \psi^{-1}_{\alpha,y}$ and note that $\tau_{\beta\alpha}(y) \in K$ according to the equation above. Also, because $\theta_{\beta\alpha}$ is continuous, so is $\tau_{\beta\alpha}$.

Hence we have shown that $\pi' : Y \times_X E \to Y$ is a fibre bundle with coordinate functions ψ_α and transition functions $\tau_{\alpha\beta}$.

Now it remains to show that f' is a bundle map. For this, let $f^{-1}U_\alpha$ and $f^{-1}U_\beta$ be coordinate neighbourhoods in Y and for $y \in f^{-1}U_\alpha \cap f^{-1}U_\beta$ define

$$f_{\beta\alpha}(y) = \phi_{\beta,f(y)} \circ f' \circ \psi^{-1}_{\alpha,y}.$$

Then

$$\begin{aligned} f_{\beta\alpha}(y)(b) = \phi_{\beta,f(y)} \circ f' \circ \psi^{-1}_{\alpha,y}(y) &= \phi_{\beta,f(y)} \circ f'(y, \phi^{-1}_{\alpha,f(y)}(b)) \\ &= \phi_{\beta,f(y)} \circ \phi^{-1}_{\alpha,f(y)}(b) = \theta_{\beta\alpha}(f(y))(b) \in K \end{aligned}$$

and

$$f_{\alpha\beta} : f^{-1}U_\alpha \cap f^{-1}U_\beta \to K$$

is continuous. Therefore f' is a bundle map. □

Definition 6. *The bundle obtained from the fibre product is called the* induced bundle *of $\pi : E \to X$ via $f : Y \to X$. It is denoted by $f^*\xi$ or f^*E.*

4 G-vector bundles

Until this, the fibre of a bundle has been just some effective K-space with no additional structure required. One possibility is to demand the fibre to be a vector space. This leads to the definition of a vector bundle:

Definition 7. *Let E and X be topological spaces with $\pi : E \to X$ a continuous mapping. If it holds for every $x \in X$ that*

1. *there exists an open neighbourhood U of x and a homeomorphism $\phi : \pi^{-1}U \to U \times \mathbb{R}^n$ with the familiar property $p \circ \phi = \pi$,*
2. *for any $y \in U$, the homeomorphism ϕ restricted to $\pi^{-1}(y)$ gives a linear isomorphism*
$$\phi|_{\pi^{-1}(y)} : \pi^{-1}(y) \xrightarrow{\simeq} \{y\} \times \mathbb{R}^n,$$

then (E, π, X) is called a vector bundle.

The sets $\pi^{-1}(y)$ are called fibers as with fibre bundles. We just demand each fiber to have a n-dimensional vector space structure, inherited from the coordinate chart homeomorphism.

Proposition 8. *A vector bundle is a fibre bundle with fibre $\mathbb{R}^n$ and structure group $GL(n, \mathbb{R})$.*

Proof. The claim is quite obvious but it is worth checking that $GL(n, \mathbb{R})$ is really the structure group.

For this, suppose U_α and U_β are overlapping coordinate charts and $x \in U_\alpha \cap U_\beta$. Define $\phi_{\alpha,x}$ and $\phi_{\beta,x}$ as usual and note that now they are linear isomorphism. Then

$$\theta_{\beta\alpha}(x) = \phi_{\beta,x} \circ \phi_{\alpha,x}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

is a linear isomorphism as well, whence we have $\theta_{\beta\alpha}(x) \in GL(n, \mathbb{R})$. The mapping $\theta_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ is essentially the same as the restriction of the continuous mapping

$$\phi_\beta \circ \phi_\alpha^{-1}|_{(U_\alpha \cap U_\beta) \times \mathbb{R}^n} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

on the set $\{x\} \times \mathbb{R}^n$, whence it is seen to be continuous. Therefore $GL(n, \mathbb{R})$ is the structure group. □

With vector bundles, we use the same terminology as with fibre bundles, so that E is called the total space, X the base space etc.

Next we define an important class of vector bundles. The difference is that we have another topological group action involved.

Definition 8. *Suppose $\pi : E \to X$ is a vector bundle ξ with E and X being G-spaces and π a G-map for some topological group G. We call ξ a G-vector bundle, if for every $g \in G$ and $x \in X$ the action map*

$$g : \pi^{-1}(x) \to \pi^{-1}(gx)$$

is a linear isomorphism.

Let us now classify some of the mappings between G-vector bundles. Suppose $\pi : E \to X$ and $\pi' : E' \to X'$ are two G-vector bundles with such G-maps $f : X \to X'$ and $\bar{f} : E \to E'$ that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ \downarrow \pi & & \downarrow \pi' \\ X & \xrightarrow{f} & X' \end{array}$$

If $\bar{f}$ is linear on the fibres $E_x = \pi^{-1}(x)$, i.e. if

$$\bar{f}|_{E_x} : E_x \to E'_{f(x)}$$

is linear for all $x \in X$, then $\bar{f}$ is a *G-bundle homomorphism.* If the restrictions are linear isomorphisms, then $\bar{f}$ is called a *G-bundle map.* In the case $X = X'$ and $f = \mathrm{Id}$, a G-bundle map $\bar{f}$ is called a *G-bundle isomorphism.*

If $\pi : E \to X$ is a G-vector bundle and $f : Y \to X$ is a G-map, take the fibre product

$$f^*E = Y \times_X E = \{(y, z) \mid f(y) = \pi(z)\},$$

which is a G-set under the diagonal action, and note that the restrictions of the natural projections are G-maps. Furthermore, we have the following property.

Proposition 9. *The construction $\pi' : f^*E \to Y$ is a G-vector bundle and $f' : f^*E \to E$ is a G-bundle map.*

Proof. Exercise, see Proposition 7. □

If $\pi : E \to X$ is a G-vector bundle and Y a G-subspace of the base-space, then one can define the *restriction* of $\pi : E \to X$ in Y in a natural way, by taking the induced bundle via the inclusion $Y \hookrightarrow X$. This is denoted by $\pi : E|Y \to X$ where $E|Y = \pi^{-1}Y$

The restriction of a G-vector bundle comes from a G-subspace of the base-space. If we start with a G-subset of the total space instead, we can define another structure as follows: Let $\pi : E \to X$ be a G-vector bundle. If a G-subspace $E' \subset E$ and the restricted mapping $\pi' = \pi|E'$ satisfy

1. $E'_x = E' \cap E_x$ is vector subspace of E_x for every $x \in X$,
2. $\pi' : E' \to X$ is a G-vector bundle such that the vector space structure of the fibers is the same as above,

then E' is called a *G-vector sub-bundle* of E.

Example 4. *Let $\pi : E \to X$ and $\pi' : E' \to X$ be two G-vector bundles over a G-space X. Their product together with the product of the projections $\pi \times \pi' : E \times E' \to X \times X$ is a G-vector bundle, called the product of E and E' (the base spaces need not to be the same).*

Now we want to consider the fibre bundle induced by the diagonal mapping

$$d : X \to X \times X, \qquad d(x) = (x, x),$$

which is clearly a G-map. The induced bundle $\pi'' : d^*(E \times E') \to X$ is called the Whitney sum *of E and E' and denoted $E \oplus E'$. Remember that the total space of this is $\{(x,(y,y')) \in X \times (E \times E') \mid d(x) = \pi \times \pi'(y,y')\}$, whence we see that the fibers are of the form*

$$\begin{aligned}(E \oplus E')_x = \pi''^{-1}(x) = \{(x,(y,y')) \mid d(x) = \pi \times \pi'(y,y')\}\\ \simeq \{(y,y') \mid \pi(y) = \pi'(y') = x\} = E_x \times E'_x,\end{aligned}$$

where $E_x \times E'_x$ denotes the direct sum of the vector spaces E_x and E'_x.

Let us now give two statements without proving them.

Lemma 1. *Let $\pi : E \to X$ be a G-vector bundle. If E_1 and E_2 are G-vector sub-bundles such that for each x the fibre E_x is the direct sum of E_{1x} and E_{2x}, then E is isomorphic to the Whitney sum $E_1 \oplus E_2$.*

Proof. Exercise. □

The previous lemma is useful when proving the following proposition that tells us a bit more about the nature of G-vector sub-bundles.

Proposition 10. *Let $\pi : E \to X$ be a G-vector bundle and $\pi' : E' \to X$ its G-vector sub-bundle. For each x there exists an open neighbourhood U and a homeomorphism $\phi : \pi^{-1}U \to U \times \mathbb{R}^n$ such that the restriction*

$$\phi|_{\pi'^{-1}U} : \pi'^{-1}U \to U \times \mathbb{R}^m$$

is also a homeomorphism for some $m \leq n$.

Proof. Exercise. □

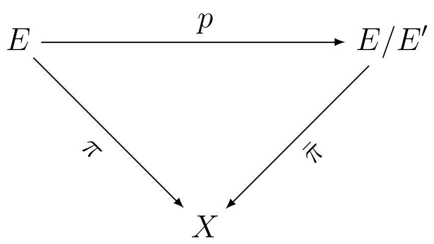
Next we begin to study quotient structures. Let $\pi : E \to X$ be a G-vector bundle and $\pi' : E' \to X$ its G-vector sub-bundle. Define a relation $\sim$ in E by putting

$$z \sim z' \qquad \Longleftrightarrow \qquad \pi(z) = \pi(z') \quad \text{and} \quad z - z' \in E'_{\pi(z)}$$

Note that $\sim$ is clearly reflexive and symmetric. It is an equivalence relation because transitivity holds too: $z - z'' = z - z' + z' - z'' \in E'_{\pi(z)} = E'_{\pi(z')}$ if $z - z' \in E'_{\pi(z)}$ and $z' - z'' \in E'_{\pi(z')}$ because these are vector spaces. Hence we can define the quotient space $E/\sim$, denoted E/E'.

What happens here is that we say two elements are equivalent if they lie in the same fiber and are equivalent in the usual sense with respect to the vector subspace of the fiber.

If p is the natural projection $E \to E/E'$ that takes $z \mapsto z + E'_{\pi(z)}$, then p coinduces a topology to E/E'. With respect to this topology, we find a continuous mapping $\bar{\pi}$ such that the following diagram commutes



It can be shown that in fact $\bar{\pi} : E/E' \to X$ is a G-vector bundle, called the *quotient G-vector bundle* of E by E'. In particular, the fibers of this bundle are easy to find:

$$(E/E')_x = \bar{\pi}^{-1}(x) = p\pi^{-1}(x) = pE_x = E_x/E'_x.$$

5 The classification of G-vector bundles over G/H

For this last section, remember the definitions of cross-sections and representation spaces:

If $f : X \to Y$ and $s : Y \to X$ are continuous mappings such that f is surjective and $f \circ s = \mathrm{Id}_Y$, then s is called a *cross-section* of f. If this happens only locally, i.e. each $y \in Y$ has an open neighbourhood U and a cross-section $s_U : U \to X$ for the restriction $f|_{f^{-1}U}$, then f is said to have a *local cross-section.*

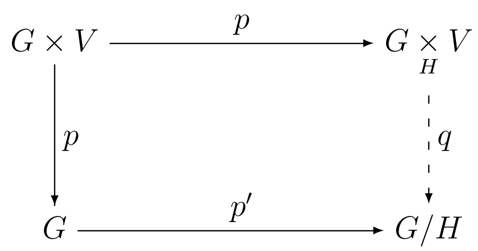
When a group G acts on a vector space V such that every $g \in G$ acts linearly, then V together with the action of G is called a G-representation space. The idea here is that every element of the group is seen as a transformation of the vector space. For example, when the base space of a G-vector bundle consists of only one point, then the total space is a G-representation space.

Now suppose G is a topological group, H a closed subgroup of G, and let $\pi : E \to G/H$ be a G-vector bundle. We can easily show that the fibre $\pi^{-1}(eH)$ is a H-representation space: by assumption the action mappings $g : \pi^{-1}(xH) \to \pi^{-1}(gxH)$ are linear isomorphism. For $h \in H \subset G$ they get the form

$$h : \pi^{-1}(eH) \to \pi^{-1}(eH)$$

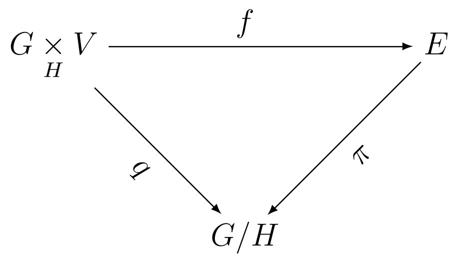
since eH is fixed by H and hence $\pi^{-1}(eH)$ is H-invariant. This is to say that the fibre $\pi^{-1}(eH)$ is a H-representation space.

Denote $V = \pi^{-1}(eH)$ and let H act on $G \times V$ by $h.(g, v) = (gh^{-1}, hv)$ and on G by $h.g = gh^{-1}$. Then the projection $G \times V \to G$ is clearly a H-map and hence induces a map $q : G \underset{H}{\times} V \to G/H$ between the orbit spaces:



This leads to the following theorem.

Proposition 11. *If we define a map $f : G \underset{H}{\times} V \to E$ by $f([g, v]) = gv$, then f is a G-homeomorphism and the following diagram commutes:*



Proof. We are going to show the homeomorphy directly, i.e. finding a continuous inverse f' for f.

First note that f itself is a well-defined and continuous G-map: if $[g, v] = [g', v']$, then $(g, v) = h.(g', v')$ and hence $g = g'h^{-1}$ and $v = hv'$ for some $h \in H$. Now $f([g, v]) = gv = g'h^{-1}hv' = g'v' = f([g', v'])$, whence f is well-defined. Continuity, in turn, follows from the continuity of the G-action. Finally, if $g' \in G$, then $f(g'.[g, v]) = f([g'g, v]) = g'gv = g'f([g, v])$, whence f is a G-map.

Let us now construct the mapping f'. Take $z \in E$ and suppose $\pi(z) = gH$ for $g \in G$. Put $v = g^{-1}z$ and note that since π is a G-map, we get

$$\pi(v) = g^{-1}\pi(z) \in g^{-1}(gH) = H = eH,$$

which is to say that $v \in \pi^{-1}(eH) = V$. Now define a G-map $f' : E \to G \underset{H}{\times} V$ by putting $f'(z) = [g, v]$. At least this a very good candidate for the inverse of f, because

$$(f \circ f')(z) = f([g, v]) = gv = g(g^{-1}z) = z$$

and

$$(f' \circ f)[g, v] = f'(gv) = [g, g^{-1}gv] = [g, v].$$

It remains to show that f' is well-defined and continuous.

For well-definement, suppose we have another representation $\pi(z) = g'H$ and choose $v' = g'^{-1}z$. Since $g^{-1}g'H = H$ we find an element $h \in H$ such that $g^{-1}g' = h$. Then clearly

$$g'h^{-1} = g'g'-1(g^{-1})^{-1} = g$$

and

$$hv' = g^{-1}g'g'^{-1}z = g^{-1}z = v,$$

showing that $[g', v'] = [g'h^{-1}, hv'] = [g, v]$. This is to say that f' is well-defined.

Next step is to show that f' is continuous, which requires a little more work with diagrams.

Let us start with the pull-back diagram

$$\begin{array}{ccc} G \times_{G/H} E & \xrightarrow{\quad p' \quad} & E \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ G & \xrightarrow{\quad p \quad} & G/H \end{array}$$

where $G \times_{G/H} E = \{(g, z) \in G \times E \mid p(g) = \pi(z)\}$. Remember that since p is surjective and open, so is p' too.

Define $f_1 : G \times E \to G \times E$ by putting $f_1(g, z) = (g, g^{-1}z)$. Then f_1 is continuous, and if $(g, v) \in G \times V$ is arbitrary, then $v \in \pi^{-1}H$ and $p(g) = gH = g(vH) = g\pi(v) = \pi(gv)$, meaning that $(g, gv) \in G \times_{G/H} E$ and $f_1(G \times_{G/H} E) = G \times V$.

Choose q_1 to be the projection $G \times V \to G \underset{H}{\times} V$ and continue to the following diagram

$$\begin{array}{ccc} G \times_{G/H} E & \xrightarrow{\quad f_1 \quad} & G \times V \\ \Big\downarrow p' & & \Big\downarrow q_1 \\ E & \xrightarrow{\quad f' \quad} & G \underset{H}{\times} V \end{array}$$

This shows that f' is continuous, because q_1 and f_1 are continuous, p' is open and surjective, and, what's most important, the diagram commutes:

$$(q_1 \circ f_1)(g, z) = q_1(g, g^{-1}z) = [g, g^{-1}z]$$

and

$$(f' \circ p')(g, z) = f'(z) = [g, g^{-1}z]$$

Now we have shown that the G-mapping f' is continuous inverse for the continuous G-mapping f, whence f is a G-homeomorphism.

□

Proposition 12. *Let G be a topological group and H its closed subgroup. Then the projection $p : G \to G/H$ has a local cross-section if and only if $p : G \to G/H$ is a principal H-bundle.*

Proof. For a principal bundle $p : G \to G/H$, one can show that the mappings $s_\alpha : U_\alpha \to \pi^{-1}U_\alpha$, $s_\alpha(x) = \phi_\alpha^{-1}(x, e)$ form a local cross-section for p.

For the other direction, assume that p has a local cross-section. Then the neutral element eH has an open neighbourhood U with a continuous mapping $s : U \to G$ satisfying $p \circ s = \mathrm{Id}_U$. We will construct the fibre bundle structure for $p : G \to G/H$ using this mapping.

Let us begin by defining the following mappings

$$\phi : p^{-1}U \to U \times H, \qquad \phi(g) = (p(g), (s(p(g)))^{-1}g)$$

and

$$\phi' : U \times H \to p^{-1}U, \qquad \phi'(u, h) = s(u)h.$$

A direct calculation gives

$$\begin{aligned} (\phi \circ \phi')(u, h) &= \phi(s(u)h) = (p(s(u)h), ((s(p(s(u)h)))^{-1}s(u)h)) \\ &= (p(s(u)), ((s(p(s(u))))^{-1}s(u)h)) = (u, s(u)^{-1}s(u)h) = (u, h) \end{aligned}$$

and

$$(\phi' \circ \phi)(g) = \phi'(p(g), (s(p(g)))^{-1}g) = s(p(g))(s(p(g)))^{-1}g = g,$$

whence ϕ and ϕ' are inverses for each other. Since both are continuous, they are homeomorphisms too.

Since U is a neighbourhood for the neutral element eH, we know that g_0U is a neighbourhood for an arbitrary element $g_0H \in G/H$. We can define a cross-section $s_{g_0} : g_0U \to G$ for this neighbourhood in a natural way by putting

$$s_{g_0}(x) = g_0 s(g_0^{-1}x), \qquad x \in g_0U.$$

Using the same idea as above, we continue to define mappings

$$\phi_{g_0} : p^{-1}(g_0U) \to g_0U \times H, \qquad \phi_{g_0}(g) = (p(g), (s_{g_0}(p(g)))^{-1}g)$$

and

$$\phi'_{g_0} : g_0U \times H \to p^{-1}(g_0U), \qquad \phi'_{g_0}(x,h) = s_{g_0}(x)h = g_0 s(g_0^{-1}x)h.$$

One can see clearly that the calculations done above with ϕ and ϕ' go also with the mappings ϕ_{g_0} and ϕ'_{g_0}, whence they are each other's inverses and thus homeomorphisms.

Now we have managed to define the coordinate functions and neighbourhoods quite easily using the local cross-section $s : U \to G$. The next step is to find the transition functions. It turns out that this is easy and straightforward too.

If $g_0, g_1 \in G$ and the charts g_0U and g_1U are overlapping such that there exists some $x \in g_0U \cap g_1U$, then

$$\begin{aligned}(\phi_{g_1} \circ \phi'_{g_0})(x,h) &= \phi_{g_1}(s_{g_0}(x)h) \\ &= (p(s_{g_0}(x)h), (s_{g_1}(p(s_{g_0}(x)h)))^{-1}s_{g_0}(x)h) \\ &= (x, s_{g_1}(x)^{-1}s_{g_0}(x)h).\end{aligned}$$

In the beginning of this material we mentioned that the equation above is often used to define transition mappings, which is what we will do now. Note that since $s_{g_0}(x)H = s_{g_1}(x)H$, we have $s_{g_1}(x)^{-1}s_{g_0}(x) \in H$, and hence we are able to define

$$\theta_{g_1g_0} : g_0U \cap g_1U \to H, \qquad \theta_{g_1,g_0}(x) = s_{g_1}(x)^{-1}s_{g_0}(x).$$

The mapping $\theta_{g_1g_0}$ so obtained is continuous since it is a composition of three continuous mappings.

Therefore we have shown that $p : G \to G/H$ is a principal fibre bundle with structure group and fibre H, coordinate functions ϕ_{g_0} and transition functions $\theta_{g_1 g_0}$.

□

The previous proposition has its consequenses in the theory of Lie groups, since one can show that whenever G is a Lie group and H its closed subgroup, the projection $p : G \to G/H$ has a local cross-section.

The following statement yields a classification for G-vector bundles over G/H.

Proposition 13. *Suppose $p : G \to G/H$ has a local cross-section. If V is an H-representation space, then $q : G \underset{H}{\times} V \to G/H$ is a G-vector bundle, where q is as in the discussion before Proposition 11. Furthermore, $q^{-1}(eH)$ is isomorphic to V as H-representation spaces.*

Proof. Since $p : G \to G/H$ has a local cross-section, it is a principal bundle. Eventhough H doesn't necessarily act effectively on V, we can consider the associated bundle $q : G \underset{H}{\times} V \to G/H$. By making use of the H-representation, we can think that the transition mappings of p go into $GL(n, \mathbb{R})$, whence the bundle becomes a vector bundle. Then $G \underset{H}{\times} V$ is a G-space by $g.[g', v] = [gv', v]$ and, furthermore, one can show that that the mappings

$$g : q^{-1}(g_0 H) \to g g_0 H$$

are linear, whence the bundle $q : G \underset{H}{\times} V \to G/H$ is actually a G-vector bundle. The last claim follows from the fact that the action of H on $q^{-1}(eH)$ coincides with the action of H on V:

$$h.[e, v] = [h, v] = [e, hv].$$

□

Corollary 1. *If $p : G \to G/H$ has a local cross-section, then the assignment $V \mapsto G \underset{H}{\times} V$ gives a one-to-one correspondence between isomorphism classes of H-representation spaces and isomorphism classes of G-vector spaces over G/H.*

Corollary 2. *If $p : G \to G/H$ has a local cross-section, then $G \underset{H}{\times} V \to G/H$ is a fibre bundle for any effective H-space F. (The fiber of this is F and structure group H.)*