Topology I
Exercise 2, spring 2012

1 (1:7, a part). Consider the norms $|x|$ = the usual Euclidean norm, $|x|_1 = |x_1| + \cdots + |x_n|$ and $|x|_\infty = \max(|x_1|, \cdots, |x_n|)$ in $\mathbf{R}^n$. Show that

(a) $|x|_\infty \le |x| \le |x|_1 \le n|x|_\infty$ for all $x \in \mathbf{R}^n$,

(b) $|x|_1 \le \sqrt{n}|x|$ for all $x \in \mathbf{R}^n$.

Tips. At the part (a), for the middle inequality, write $x = \sum_{i=1}^n x_i e_i$ and apply the triangle inequality. At (b) apply the Schwarz's inequality to the vectors $(|x_1|, \cdots, |x_n|)$ and $(1, \cdots, 1)$.

2 (1:10). Show that the equation $\|x\| = \max\{|x_1| + |x_2|, 2|x_1|\}$ defines a norm in the plane $\mathbf{R}^2$. Draw also the unit sphere $S = \{x \in \mathbf{R}^2 \mid \|x\| = 1\}$.

Tips. Study first separately the components (defining the norm) of norms of $x+y$ and ax. At the end part the unit spheres again separately given by the components of norm, and draw a conclusion.

3. Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbf{R}^2$. Study whether the mapping

$$d(x, y) = |x_1^5 - y_1^5| + |x_2^3 - y_2^3|$$

is a metric in the set $\mathbf{R}^2$.

4. Determine the ball $B(a, 3)$ in $\mathbf{R}^2$, when $a = (-1, 1)$ and the used metric is $d(x, y) = |x_1 - y_1| + 2|x_2 - y_2|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. It is not necessary to show that d is a metric, but also you can do it. (How most easily?)

A tip. Set first $a = (0, 0)$ and then transfer.

5. Let $X = \{x \in \mathbf{R} \mid x > 0\}$, and define in the set $X \times X$ a real valued mapping

$$d(x, y) = \Big|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}\Big|, \quad x, y \in X.$$

Show that d is a metric in the set X.

6. (2:13) Let $E = raj([0, 1], \mathbf{R})$ equipped with the supnorm. Determine the distance $d(A, B)$ between its subsets $A = \{f_n : [0, 1] \to \mathbf{R} \mid f_n(x) = x^n,\, n \in \mathbf{N}\}$ and $B = \{f : [0, 1] \to \mathbf{R} \mid f \textit{ is a constant function}\}$.

A tip. Have a look at a certain constant function against that the other ones are compared .