Topology I
Exercise 9, spring 2012

1. Let (f_n) be a sequence of continuos functions $f_n : [a,b] \to \mathbf{R}$ that converge uniformly on $[a,b]$ to a function $f : [a,b] \to \mathbf{R}$. Show that then

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

Why do the integrals exist? Briefly, does the corresponding result hold to derivatives?

2. Let $A = \{(x,y) \in \mathbf{R}^2 \mid 0 < y < x^2\}$ and $B = \mathbf{R}^2 \setminus A$. Obviously $\mathbf{0} = (0,0) \in \bar{A}$ and $\mathbf{0} \in \bar{B}$ (an illustrating figure for yourself). Define a function $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(z) = f(x,y) = \begin{cases} 1, & \text{when } z = (x,y) \in A, \\ 0, & \text{when } z = (x,y) \in B. \end{cases}$$

(a) Give $\lim_{z\to\mathbf{0},\, z\in A} f(z)$ and $\lim_{z\to\mathbf{0},\, z\in B} f(z)$ taken through A and B.

(b) What conclusion do those limits offer, if continuity of f at $\mathbf{0}$ is asked?

(c) Show that $\lim_{z\to\mathbf{0},\, z\in L} f(z) = 0$ for all straight line L passing through the origin.

3 (12:11). Let X be a complete metric space and $f : X \to Y$ bilipschitz. Show that the image set fX is complete and thus closed in Y.

4 (12:7). (a) Let X be a complete metric space and $A_1 \supset A_2 \supset \cdots$ a nested sequence of closed nonempty subsets of it such that the diameters $d(A_n)$ converge to zero. Show that the intersection of sets A_n has precisely one point then.

(b, adaption) Give an example of subsets U_n in $\mathbf{R}$ that are like A_n in item (a), but are open instead of being closed, and however their intersection is empty.

A tip. (a) Choose for every $n \in \mathbf{N}$ a point $x_n \in A_n$ and consider the sequence (x_n). Completeness of X is necessary here.

5 (12:14). Let $(E, \|*\|)$ be a complete normed space, that is a Banach space, and let $f : E \to E$ be a contraction. Show that the equality $F(x) = x + f(x)$ defines a homeomorphism $F : E \to E$ that is bilipschitz.

Tips. Fix $y \in E$ and denote $g_y(x) = y - f(x)$. Show that the mapping $g_y : E \to E$ has precisely one fixed point $G(y)$, when they together define the mapping $G : E \to E$, $y \mapsto G(y)$. Then show that $F \circ G = G \circ F = id_E$ and that F is bilipschitz. Pay special attention to the "left side" inequality $m\|x - z\| \le \|F(x) - F(z)\|$ for all $x, z \in E$, where the constant must satisfy $m > 0$.

6 (12:15, a part). Study whether the following functions $f : \mathbf{R} \to \mathbf{R}$ are uniformly continuous on $\mathbf{R}$:

$$(a) \quad f(x) = \frac{x}{1+x^2}, \quad (b) \quad f(x) = x^{1/3}.$$

A tip. The mean value theorem can be useful.