Topology I
Exercise 8, spring 2012

1 (11:2). Let X be a metric space, $A \subset X$ and (x_k) a sequence in A (that is $x_k \in A$ for all $k \in \mathbf{N}$). Show that any accumulation value of the sequence belongs to the closure $\bar{A}$.

2. Study whether the following sequence (x_k) converges in $\mathbf{R}^2$. If it does it , give also a limit point. If not, does the sequence have any convergent subsequence (and thus any accumulation value)? A brief answer.
(a) $x_k = \big((1/5)^k, 1^k\big)$, (b) $x_k = (2^{-k}, (-1)^k)$, (c) $x_k = \big(k^{1/2}, (-2)^{-k}\big)$.

3. Prove indirectly (supposing the opposite) the implication (2) $\Rightarrow$ (1) of Theorem 11.8, i.e. if a function $f : X \to Y$ is sequently continuous at a point $a \in X$, so it also is continuous at that point (this result concerns metric spaces X and Y).

4. Let (x_k) and (y_k) be real number sequences such that $x_k \to 3\pi/4$ and $y_k \to \sqrt{\pi}/2$. Denote $w_k = \sin(x_k - y_k^2)$. Show exactly that $w_k \to 1 = \sin(\pi/2)$.
Tips. Use sequently continuity of the function $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x,y) = \sin(x - y^2)$, (the implication (1) $\Rightarrow$ (2) of Theorem 11.8) and write $z_k = (x_k, y_k) \in \mathbf{R}^2$, when $w_k = f(z_k)$.

5 (11:10, partly). Let $n \in \mathbf{N}$ and $f_n : \mathbf{R} \to \mathbf{R}$ be a function mapping $f_n(x) = \max\{0, x - n\}$ when $x \in \mathbf{R}$. Study whether the function sequence (f_n) converge (a) pointwise in $\mathbf{R}$, (b) uniformly in $\mathbf{R}$ (c) uniformly in the interval $]-\infty, 10^{10}]$.

As quite a difficult and theoretic one, the following problem deserves two performance points:

6. Let X be a set, d a metric there and $f : X \to X$ a bijection.
(a) Show that also $e(x,y) = d\big(f(x), f(y)\big)$ (for all $x, y \in X$) is a metric in X, and that the function $f : (X,e) \to (X,d)$ is a (bijective) isometry.
(b) Suppose more the function $f : (X,d) \to (X,d)$ is a homeomorphism. Show that $d \sim e$ then.
(c) Apply previous items to the set $\mathbf{R}$ and the metric $e(x,y) = |\sqrt[3]{x} - \sqrt[3]{y}|$ of it, and show that e is (topologically) equivalent to the usual Euclidean metric.