Topology I
Exercise 7, spring 2012

1. Is the function $f : [0,10] \to \mathbf{R}$, $f(x) = x^3 + x$, (a) Lipschitz, (b) bilipschitz, (c) an embedding?

A tip. The mean value theorem.

2. Consider the surface $A = \{(x, y, z) \in \mathbf{R}^3 \mid z = \sin x + \cos y\}$ in $\mathbf{R}^3$, equipped with the Euclidean metric. Suppose it is known that the function

$$f : A \to \mathbf{R}^2, \quad f(x, y, z) = (x, y) \quad \text{when } (x, y, z) \in A,$$

is a bijection. Show that it is a homeomorphism and thus $A \approx \mathbf{R}^2$. As known, the functions $\sin, \cos : \mathbf{R} \to \mathbf{R}$ are continuous.

3. Let $x \in X$ and $\emptyset \neq A \subset X$. The condition $d(x, A) > 0$ is a topological property (closure!). Let $f : (X, d) \approx (Y, e)$ be a homeomorphism. Show that really $e(f(x), f(A)) > 0$, if $d(x, A) > 0$.

A tip. Suitable theorems are for example 6.11 and 6.12.

Remark. Be careful in, what is or is not a topological property. For example $d(A, B) > 0$ is not.

4 (10:1). The mapping $e : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$, $e(x, y) = \sqrt{|x - y|}$, is a metric in $\mathbf{R}$ (need not to prove). Is it (a) equivalent, (b) bilipschitz equivalent with the usual Euclidean metric d in $\mathbf{R}$?

A tip. Better to study continuity and Lipschitz continuity in the identical mapping by using their original definitions.

As quite a difficult one, the following problem is worth of two points:

5 (9:5, an adaptation). Construct a homeomorphism $f : B^2 \approx \mathbf{R}^2$, where $B^2 = \{x \in \mathbf{R}^2 \mid |x| < 1\}$ is the open unit disk in the plane. Do you find the inverse mapping?

A tip. Move radially, that is, multiply a unit vector $x/|x|$, and therefore consider the corresponding situation in $\mathbf{R}$ (geometrically or you can also use the function $\tan : [0, \pi/2[\to [0, \infty[$, as well).

Remark. Everyone can participate in the compensating 1. course exam 12.3.