Topology I

Exercise 5, spring 2012

1. Study if the following subsets are closed in the Euclidean plane $\mathbf{R}^2$:

$$(a)\ A_k = \{(x,y) \in \mathbf{R}^2 \mid 1/(k{+}1) \le |(x,y)| \le 1/k\},\ \text{where}\ k \in \mathbf{N}, \quad (b)\ A = \bigcup_{k\in\mathbf{N}} A_k.$$

If not, determine the closure. Tips. Continuous functions, when considering closure, part (4) in Theorem 6.8 is useful.

2. Is the subset $A = \{(x,y) \in \mathbf{R}^2 \mid x > 0,\ y = \sin(1/x)\}$ cosed in $\mathbf{R}^2$? If not, determine its closure. Which are cluster points? Any proof is not now needed, only a short answer.

Tips. Draw a figure. Where do the points $\big(x_k, \sin(1/x_k)\big) \in A$ accumulate, when $x_k = 1\big/(\pi/2 + k\pi)$ and $k \in \mathbf{N}$ is even or odd. Bolzano.

3. (6:12) Suppose $f, g : X \to Y$ are continuous functions and $A \subset X$ such that $f|A = g|A$. Show that $f|\bar{A} = g|\bar{A}$.

4. (6:8) Let E be an inner product space and $A \subset E$. Show that the orthogonal complement $A^\perp = \{x \in E \mid \langle x, a\rangle = 0 \text{ for all } a \in A\}$ of A is closed in E. Thus $A^\perp$ is a closed subspace of E (problem 4 in the first exercise).

Tips. An inner product (function) is continuous. Represent $A^\perp$ for instance as an intersection.

5. (7:3 a part) Determine the closure of a set $A \subset B^2 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ in the Euclidian space B^2, when

$$(a)\ A = \{(x,0) \in B^2 \mid -1 < x < 1\}, \quad (b)\ A = \{(x,y) \in B^2 \mid x + y > 0\}.$$

Is A closed in B^2? A tip. Perhaps first the closure in $\mathbf{R}^2$.

6. Define the function $f : [-1,2] \to \mathbf{R}$ by the equality

$$f(x) = \begin{cases} -x + 1, & \text{when} \quad -1 \le x \le 0, \\ -2x^2 + x + 1, & \text{when} \quad 0 < x \le 1, \\ x^3 - x, & \text{when} \quad 1 < x \le 2. \end{cases}$$

Show that it is continuous. A tip. Theorem 7.13.

Remark. The first course exam 28.2., as well as the compensating one 12.3., includes Chapters 1-7 in Väisälä. Recall that a canditate can use a short abstract of size A4. Exercise 6 takes place first week of the 4. period, 12.-16.3.