Topology I
Exercise 4, spring 2012

1 (4:5). Suppose $f : X \to Y$ and $g : Y \to Z$ are Lipschitz-functions with L-constants M and N. Show that the composition function $g \circ f : X \to Z$ is MN-Lipschitz.

2. The function $f : [-2, 2] \to \mathbf{R}$ is defined by $f(x) = x^2/2 - x^3/3$. By using the mean value theorem determine $M \geq 0$ such that f is M-Lipschitz.

3. Let $(E, |*|)$ be a norm space, and let fix two points x and y in E. The equality

$$h(t) = (1 - t)x + ty \quad \text{for } t \in [0, 1]$$

defines a function $h : [0, 1] \to E$, the graph of which is the line segment connecting x ja y in E. Show that h is continuous.

4. (4:4, about). Suppose $E = C([0, 1], \mathbf{R})$ is equipped with the supnorm and the metric defined by that. For every $f \in E$ the equality

$$\alpha(f)(x) = 5xf(x) \quad \text{for } x \in [0, 1]$$

defines a function $\alpha(f) : [0, 1] \to \mathbf{R}$.
(a) Show that $\alpha(f)$ is continuous, i.e. $\alpha(f) \in E$.
(b) From part (a) it follows that we have a mapping $\alpha : E \to E$, $f \mapsto \alpha(f)$. Show that α is continuous.
(c) Show that α is Lipschitz.
A tip. In part (a) Chapter 5 in Väisälä.

5. Consider the subset

$$A = \{(x, y) \in \mathbf{R}^2 \,|\, x^2 - x < y < 2x\}$$

in the Euclidean $\mathbf{R}^2$. Show that A is an open set.
A tip. Recall continuous functions and Chapter 5.

6. Suppose $f, g : X \to \mathbf{R}$ are continuous functions. Show that then also
(a) the maximum function $h : X \to \mathbf{R}$, where $x \mapsto h(x) = \max\{f(x), g(x)\}$, is continuous,
(b) the absolute value function $|f| : X \to \mathbf{R}$, where $x \mapsto |f|(x) = |f(x)|$, is continuous.
Tips. (a) Fix $a \in X$, when you can suppose $h(a) = f(a)$, and you get a neighbourhood of a needed as an intersection of two neighbourhoods. (b) Use (a).