Topology I

Exercise 3, spring 2012

1. (2:12) Let be $E = raj([0,1], \mathbf{R})$ equipped with the supnorm and the metric induced by that. Determine $d(A)$ for $A = \{f_n : [0,1] \to \mathbf{R} \mid f_n(x) = x^n, n \in \mathbf{N}\}$.
A tip. Draw graphs. Calculate $\lim_{n\to\infty}(x - x^n)$ for $0 \le x < 1$?

2. Which of the following subsets of $\mathbf{R}^2$ are open:
(a) $A = \{(x,y) \in \mathbf{R}^2 \mid x = 2y^2\}$, (b) $B = A^c$, (c) $C = \{(x,y) \in \mathbf{R}^2 \mid \sin x^2 > y\}$?
A one word answer is enough.

3 (3:3). Let X be a metric space, $G \subset X$ open and $F \subset X$ finite. Show that the set $G \setminus F$ is open.

4. Let X be a metric space, a mapping $f : X \to \mathbf{R}$ continuous at $a \in X$ and $f(a) > 0$. Show that a has a neighbourhood $U \subset X$ such that $f(x) > f(a)/2$ for all $x \in U$.

5 (4:8, a part). Let

$$f(\mathbf{0}) = 0 \quad \text{and} \quad f(x,y) = \frac{xy^2}{x^2 + y^4} \quad \text{for } (x,y) \neq \mathbf{0}.$$

Show that the function $f : \mathbf{R}^2 \to \mathbf{R}$, defined by that, is discontinuous at the origin.

6. Consider the space of all continuous functions $f : [0,1] \to \mathbf{R}$, $E = C([0,1], \mathbf{R})$, equipped with the supnorm $\|f\|_\infty = \max\{|f(x)| : x \in [0,1]\}$ (as known, here sup is max) and the metric induced by that. Which of the following sets are open in E (arguments):
(a) $A = \{f \in E : \|f\|_\infty > 0\}$, (b) $B = \{f \in E \mid f(x) > 0 \ \forall x \in [0,1]\}$,
(c) $C = \{f \in E \mid f(1/n) > 0 \ \forall n \in \mathbf{N}\}$?
Tips. (a) A complement, (b) a continuous function $f : [0,1] \to \mathbf{R}$ attains its maximum and minimum, (c) an appropriate $f \in C$.