Topology I

Exercise 11, spring 2012

1 (14:12, adaption). Consider the subset $E = \{(x,y) \in \mathbf{R}^2 : |x| > |y|\}$ of $\mathbf{R}^2$.

(a) Is E connected? (b) Is the closure $\bar{E}$ connected?

Tips. (b) Intervals with begins at the origin, i.e. pathwise connectivity. Draw a figure for yourself.

2. Consider the subsets of $\mathbf{R}^2$ as in problem 1 of exercise 10:

$A_1 = \{(x,y) \,|\, x^2/3 + y^2 \le 4\}$, $A_2 = \{(x,y) \,|\, x^2y^2 = 1\}$, $A_3 = \{(x,y) \,|\, x^2 + y^2 < 4\}$. Which of those are connected? Domains in $\mathbf{R}^2$?

A tip. The same strategy as in problem 1.

3. Let (X,d) be a metric space and $I = [0,1]$. Let $\alpha : I \to X$ and $\beta : I \to X$ be paths such that $\alpha(1) = \beta(0)$, that is, the end point of the first one is the starting point of the second one. Construct by using α and β paths $\gamma : I \to X$ and $\eta : I \to X$ of X such that $\gamma(I) = \eta(I) = \alpha(I) \cup \beta(I)$, $\gamma(0) = \alpha(0)$ and $\gamma(1) = \beta(1)$, on the other hand $\eta(0) = \beta(1)$ and $\eta(1) = \alpha(0)$. One can say that γ goes α and β in succession and η in turn does it backwards.

A tip. Define in pieces.

4 (14:4). Let $A \subset \mathbf{R}$, $A \ne \emptyset$ and $X = A \times [0,1] \subset \mathbf{R}^2$. Let $f : X \to \mathbf{R}^2$ be a continuous function such that $f(x,0) = (0,0)$ for all $x \in A$. Show that the image set fX is connected.

Tips. We do not know the space X is connected, but we do that $[0,1]$ is. Many possibilities, Theorem 14.12 or pathwise connectivity.

5 (14:9). Show that the subset $A = \{(x,y,z) \in \mathbf{R}^3 \,|\, x + \cos y - e^y \sin z = 1\}$ of $\mathbf{R}^3$ is connected.

A tip. Theorem 14.16, a continuous function $f : \mathbf{R}^2 \to \mathbf{R}^3$ such that $Im f = A$.

6 (14:8, adaption). Let G be a domain of a metric space (X,d). Suppose subsets $A, B \subset \partial G$ are closed, nonempty and disjoint. Show that there exists a point $x \in G$ such that $d(x,A) = d(x,B)$.

A tip. Consider the continuous function $f : X \to \mathbf{R}$, $f(x) = d(x,A) - d(x,B)$, Theorem 14.19. Actually, $A \not\subset B$ and $B \not\subset A$ are enough to separate here A from B sufficiently.

Remark. The second course exam 9.5. (13-15, auditorium of Exactum) includes the chapters 8-14 of Väisälä, indeed excluding product spaces and connected components.