**Topology I**
Exercise 10, spring 2012

1 (13:3, about). Study briefly whether a set $A_k \subset \mathbf{R}^2$ is (a) compact, (b) complete, when

$A_1 = \{(x,y) \,|\, x^2/3+y^2 \le 4\}$, $A_2 = \{(x,y) \,|\, x^2y^2 = 1\}$, $A_3 = \{(x,y) \,|\, x^2+y^2 < 4\}$.

2. Let $A \neq \emptyset$ be a closed and bounded subset of $\mathbf{R}^2$. Show that there exists a point $(a,b) \in A$ such that $x^2 + 2\sin y \ge a^2 + 2\sin b$ for all $(x,y) \in A$.

A tip. Use a continuous function.

3 (13:21). Let a function $f : \mathbf{R}^n \to \mathbf{R}$ be continuous, and suppose it is uniformly continuous on the set $\mathbf{R}^n \backslash B^n$ ($B^n$ is the open unit ball). Show that $f$ is uniformly continuous on the whole $\mathbf{R}^n$. Remark. The target space of function could be any metric space.

Tips. So a large ball $A = \bar{B}(\mathbf{0}, r)$ and so a little $\delta > 0$, that $x, y \in A$ or $x, y \in \mathbf{R}^n \setminus B^n$, whenever $|x - y| < \delta$. Draw a figure for yourself.

4 (13:4, adaptation). Let $(X,d)$ be a compact space, and let $A_1 \supset A_2 \supset \cdots$ be a nested sequence of closed nonempty subsets of it.

(a) Show by a sequence, that the intersection $\cap_{n \in \mathbf{N}} A_n$ is nonempty and compact.

Remark. If additionally $d(A_n) \to 0$, so it is a singleton (see problem 4, ex. 9).

(b) Is it necessarily nonempty, if supposing compactness is dropped out?

A tip. (b) Consider $X = \mathbf{R}$ and choose appropriate nested closed intervals in it.

5. (a) Let $r > 0$, and let $A$ be such a subset of a metric space $(X,d)$, that there exists a sequence $(x_n)$ of $A$ such that $d(x_k, x_n) \ge r$ for all $k \neq n$. Show that $A$ is not compact then.

(b) Equip the space $E = C([0,1], \mathbf{R})$ of continuos functions $f : [0,1] \to \mathbf{R}$ with the supnorm $\|*\|_\infty$, $\|f\|_\infty = \sup\{|f(x)| \,:\, x \in [0,1]\}$ when $f \in E$. Show by using part (a) that the closed unit ball

$$\bar{B} = \bar{B}(\mathbf{0}, 1) = \{f \in E \,:\, \|f\|_\infty \le 1\}$$

of $E$ is not compact, although it, as known, is a closed and bounded set in $E$.

A tip. (b) A sequence of (simple) functions $f_n : [0,1] \to \mathbf{R}$, defined in pieces and alive in $]1/(n+1), 1/n[$, that is, never two alive simultaneously.

6 (13:18, deserves two points). Let $B^n$ be the open unit ball of the space $\mathbf{R}^n$ and $f : B^n \to B^n$ be a homeomorphism. Let $(x_k)$ be a sequence of points in $B^n$ such that $|x_k| \to 1$, when $k \to \infty$. Show that $|f(x_k)| \to 1$. Remark. An analogous result is valid also for the closed unit ball $\bar{B}^n$ (the Brouwer's Domain Theorem).

A tip. Indirect proof, compactness is needed.