Topology I
Exercise 1, spring 2012

1. Which of the following claims are true? A one word answer is completely enough. $A, B, X, Y, \cdots$ sets, $f : X \to Y$ a function.

$$(a) \quad f^{-1} \cup_{i\in I} B_i = \cup_{i\in I} f^{-1} B_i, \quad (b) \quad f \cap_{i\in I} A_i = \cap_{i\in I} f A_i,$$
$$(c) \quad f^{-1} f A = A, \quad \text{when } f \text{ is injektive}, \quad (d) \quad \cap_{x>0} [x, 2x] = \emptyset.$$

2. Let $x = (1, -2, 1)$ and $y = (2, -2, -1)$ be vectors of the Euclidean space $\mathbf{R}^3$ and $a = -3$. Calculate
(a) $a(x - y)$, (b) $|a||x - y|$, (c) $|a|(|x| - |y|)$, (d) $a(x \cdot y)$, (e) $|a||x||y|$, (f) $x \cdot |a|y$, where the usual dot product and usual Euclidean norms in $\mathbf{R}^3$ and $\mathbf{R}$ are used.

3. Let $D \neq \emptyset$ be a set. Show that the set $raj(D, \mathbf{R})$ of all bounded functions $D \to \mathbf{R}$ is a subspace of the vector space $F(D, \mathbf{R})$ of all functions $D \to \mathbf{R}$.

4 (Väisälä 1:4). Let E be an inner product space and $\langle \cdot, \cdot \rangle$ its inner product. *An ortocomplement* of a subset $A \subset E$ is the set

$$A^\perp = \{x \in E \mid \langle x, y \rangle = 0 \text{ for all } y \in A\}.$$

Show that the set $A^\perp$ is a subspace of the vector space E.

5 (Väisälä 1:6, a part). Let E be an inner product space and $x, y \in E$. Prove the equality

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2.$$

6. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbf{R}^3$. Study which ones of the inner product postulates are satisfied by the real valued product

$$\langle x, y \rangle = x_1 y_3 + x_2 y_2 + x_3 y_1$$

in $\mathbf{R}^3$. Is it an inner product there?