Matematiikan ja tilastotieteen laitos
Topics in geometric Fourier analysis
Exercise 6

1. Show that if $A \subset [0,1]$ and $\mathcal{L}^1(A) > 99/100$, then A contains some points $s, s+d, s+2d$ with $d > 1/10$. How much smaller can you make 99/100 and still prove this?

2. Prove that the Hausdorff dimension is at most the lower Minkowski dimension for any set in $\mathbb{R}^n$.

3. Give an example to show that Hausdorff dimension can be strictly less than lower Minkowski dimension.

4. Show that if there exists a modified Besicovitch set, as in the proof of Theorem 9.7, of Minkowski dimension d, then there exists a Besicovitch set of Minkowski dimension d.

5. The lower packing dimension $\dim_P A$ of $A \subset \mathbb{R}^n$ can be defined as

$$\dim_P A = \inf\{\sup_i \dim_M C_i : A \subset \cup_i C_i, C_i \text{ compact}\}.$$

Prove that if $\dim_M B \geq d$ for every Besicovitch set $B \subset \mathbb{R}^n$, then $\dim_P B \geq d$ for every Besicovitch set $B \subset \mathbb{R}^n$. Recall that Besicovitch are defined to be compact.
Hint: Baire category theorem might help.

6. Complete the proof of Theorem 9.3 by doing the exercise at the last line of its proof.