Matematiikan ja tilastotieteen laitos
Topics in geometric Fourier analysis
Exercise 3
17.2.2012

1.Let $\mu_j, j = 1, 2$, be finite Borel measures with compact support in $\mathbb{R}^{n_j}$ and let $\mu = \mu_1 \times \mu_2$ be the product measure in $\mathbb{R}^{n_1+n_2}$. Prove that the restriction estimate

$$||\hat{f}||_{L^p(\mu)} \leq C_{p,q}||f||_q, f \in \mathcal{S}(\mathbb{R}^{n_1+n_2}),$$

is equivalent with that the analogous estimates hold for both μ_1 and μ_2 (with the same p and q). Conclude that the restriction theory for the sphere in $\mathbb{R}^n$ is essentially the same as for the cylinder $S^{n-1} \times [0, 1]$ in $\mathbb{R}^{n+1}$.

2. Let μ be a finite Borel measure with compact support in $\mathbb{R}^n$. Prove that if the restriction estimate

$$||\hat{f}||_{L^p(\mu)} \leq C_{p,q}||f||_q, \ f \in \mathcal{S}(\mathbb{R}^n),$$

holds, then it holds for all p_1, q_1 with $1 \leq p_1 \leq p$ and $1 \leq q_1 \leq q$.

Hint: Use the fact that $\hat{f} = \widehat{f * \phi}$ on the support of μ if $\phi \in \mathcal{S}$ and $\hat{\phi}$ equals 1 on the support of μ.

3.Let μ be a finite Borel measure with compact support in $\mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^n$ an affine bijection ($Tx = Lx + a$ with L a linear bijection and $a \in \mathbb{R}^n$). Let ν be the push-forward of μ under T; $\int g d\nu = \int g \circ T d\mu$. Show that if the restriction estimate as in Exercise 2 holds for μ it holds also for ν.

4. Let $R \subset \mathbb{R}^2$ be a rectangle with side-lengths r_1 and r_2 and ϕ a C^∞-function with support in R. Find a 'dual' rectangle $\tilde{R}$ of R with side-lengths $1/r_1$ and $1/r_2$ such that $\hat{\phi}$ decays very fast (what should this mean?) outside $\tilde{R}$.