One dimensional diffusions and boundary conditions

1 Tanaka's formula: local time of the Wiener process

$$|w_t| = \int_0^t dw_s \operatorname{sgn} w_s + \ell_{t;w}^{(0)} \tag{1.1a}$$

$$\ell_{t;w}^{(0)} = \int_0^t ds\, \delta(w_s) \tag{1.1b}$$

Ito lemma

$$d|w_t| = dw_t\, \partial_{w_t}|w_t| + \frac{dt}{2}\, \partial_{w_t}^2 |w_t| = dw_t \operatorname{sgn} w_t + dt\, \delta(w_t) \tag{1.2}$$

Note that

$$\operatorname{sgn} x = H(x) - H(-x) \tag{1.3}$$

since

$$\frac{d}{dx} H(x) = \delta(x) \tag{1.4}$$

we have

$$\frac{d}{dx} \operatorname{sgn} x = 2\, \delta(x) \tag{1.5}$$

1.1 Conditional expectation of the local time

The expectation value of the local time spent at x by a Wiener process starting at the origin at time zero is

$$\mathrm{E}_{0,0} \ell_{t;w}^{x} = \int_0^t ds \int_{\mathbb{R}} \delta(y-x)\, \frac{e^{-\frac{y^2}{2\,s}}}{(2\,\pi\,s)^{1/2}} = \int_0^t ds\, \frac{e^{-\frac{x^2}{2\,s}}}{(2\,\pi\,s)^{1/2}} \tag{1.6}$$

Let us observe that

$$\mathrm{E}_{0,0} \ell_{\lambda^2\,t;w}^{\lambda\,x} = \int_0^{\lambda^2 t} ds\, \frac{e^{-\frac{\lambda^2 x^2}{2\,s}}}{(2\,\pi\,s)^{1/2}} = \lambda \int_0^t ds\, \frac{e^{-\frac{x^2}{2\,s}}}{(2\,\pi\,s)^{1/2}} = \lambda\, \mathrm{E}_{0,0} \ell_{t;w}^{x} \tag{1.7}$$

Let us choose now

$$\lambda = \frac{1}{\sqrt{t}} \tag{1.8}$$

we have then

$$t^{1/2}\,\mathrm{E}_{0,0}\ell_{1;w}^{\frac{x}{\sqrt{t}}} = \mathrm{E}_{0,0}\ell_{t;w}^{x} \tag{1.9}$$

whence we see that if we take the limit of t tending to infinity for *fixed* x we obtain

$$\mathrm{E}_{0,0}\ell_{t;w}^{x} \overset{t\uparrow\infty}{\sim} t^{1/2}\,\mathrm{E}_{0,0}\ell_{1;w}^{0} \tag{1.10}$$

The natural tool to study scaling behavior is the Mellin transform

$$(\widetilde{\mathrm{E}_x\ell_{t;w}^{0}})(z) = \int_{\mathbb{R}} \frac{dw}{w}\frac{1}{w^z}\int_0^{w\,t} ds\,\frac{e^{-\frac{x^2}{2\,s}}}{(2\,\pi\,s)^{1/2}} = \frac{t^z}{z}\int_0^{\infty} ds\,\frac{e^{-\frac{x^2}{2\,s}}}{(2\,\pi\,s)^{1/2}s^z} \tag{1.11}$$

It yields for $\Re z > 1/2$

$$(\widetilde{\mathrm{E}_x\ell_{t;w}^{0}})(z) = \frac{t^{1/2}\,\Gamma\left(z-\frac{1}{2}\right)}{(2\,\pi)^{1/2}\,z}\left(\frac{2\,t}{x^2}\right)^{z-\frac{1}{2}} \tag{1.12}$$

We can extricate the asymptotic behavior for $\tau\uparrow\infty$ by using Cauchy theorem to shift the integration contour to the left. We get into

$$\mathrm{E}_x\ell_{t;w}^{0} \sim \left(\frac{2\,t}{\pi}\right)^{1/2} \tag{1.13}$$

1.2 Interpretation

Let ξ_t be a differentiable trajectory and suppose

$$H = \{t\colon\,\xi_t = x\} \tag{1.14}$$

is a countable set then

$$\int_0^t ds\,\delta(\xi_s - x) = \sum_{t_i\in H}\int_0^t ds\,\delta\left(\frac{d\xi_s}{ds}(s-t_i)\right) = \sum_{t_i\in H}\frac{1}{\left|\frac{d\xi_s}{ds}(t_i)\right|} \tag{1.15}$$

Hence in order to count the number of times the process hits x we need to compute

$$\int_0^t ds\,\left|\frac{d\xi_s}{ds}\right|\delta(\xi_t - x) = \sum_{t_i\in H} 1 \tag{1.16}$$

Thus we have

$$\nu_{t;\xi}^{(x)} = \frac{1}{t}\int_0^t ds\,\left|\frac{d\xi_s}{ds}\right|\delta(\xi_t - x) \tag{1.17}$$

is the fraction of the time horizon t spent by the process in x. Hence in order to generalize the concept of local time to arbitrary diffusion we need to make sense of the Jacobian term in (1.17). Before turning to this task, let us also observe that any time average

$$\bar{f}_t = \frac{1}{t}\int_0^t ds\,f(\xi_s) \tag{1.18}$$

can be re-written as

$$\bar{f}_t = \int_{\mathbb{R}} dx\,\mu_{t;\xi}(x)\,f(x) \tag{1.19}$$

with

$$\mu_{t;\xi}(x) = \frac{1}{t}\int_0^t ds\,\delta(\xi_s - x) \tag{1.20}$$

2 Time change and local time for local martingales

Let

$$a \colon \mathbb{R} \mapsto \mathbb{R}_+ \tag{2.1}$$

strictly positive and satisfying the hypotheses of existence and uniqueness for stochastic differential equations. Consider the local martingale

$$d\xi_t = a(\xi_t)\, dw_t \tag{2.2a}$$

$$\xi_0 = x_o \tag{2.2b}$$

Proposition 2.1. *Let ζ_t the stochastic process solution of (2.2). Then*

$$\tau = \int_0^t dt_1\, a^2(\xi_{t_1}) \tag{2.3}$$

is a monotonically growing random process and we have

$$\xi_t = x_o + \tilde{w}_\tau \tag{2.4}$$

where $\tilde{w}$ is a Wiener process

Proof.
We wish to define a new time

$$\tau = f(t, \{\xi_t\})$$

such that

$$\tilde{w}_\tau = \xi_t$$

defines a Wiener process. Since arbitrary increments must satisfy

$$\mathrm{E}(\tilde{w}_{\tau_2} - \tilde{w}_{\tau_1})^2 = \mathrm{E}(\tau_2 - \tau_1)$$

then, for any realization of the process the identity

$$w_{\tau(t+\mathrm{d}t)} - w_{\tau(t)} = \sqrt{\frac{\mathrm{d}\tau}{\mathrm{d}t}}\mathrm{d}w_t = \mathrm{d}\xi_t$$

must hold true. Thus, we obtain

$$\frac{d\tau}{\mathrm{d}t} = a^2(\tilde{w}_\tau)\ >\ 0 \tag{2.5}$$

□

We can, thus, define the local time for the process specified by (2.2) by time-re-parametrization. Namely we know that (2.3) maps ξ_t into a Wiener process and by Tanaka's formula we know the expression of the local time for such Wiener process. By inverting the mapping we can identify the local time of ξ_t. Explicitly

$$\ell^{(x)}_{\tau;\tilde{w}} := \int_0^\tau ds\, \delta(w_s - x) \tag{2.6}$$

if we set

$$w_\tau = \xi_t - x_o \tag{2.7}$$

we obtain

$$\ell^{(x)}_{\tau;\tilde{w}} = \ell^{(x)}_{t;\xi - x_o} = \int_0^t ds\, a^2(\xi_s)\, \delta(\xi_s - x - x_o) \tag{2.8}$$

which defines the local time for the process (2.2)

$$\ell^{(x)}_{t,\xi} = \int_0^t ds\, a^2(\xi_s)\, \delta(\xi_s - x) \tag{2.9}$$

2.1 Relation to the quadratic variation of the local martingale

We observe that if ξ_t solves (2.2), the quadratic variation of the differential is

$$\langle d\xi_s, d\xi_s \rangle = a^2(\xi_s)\, dt \tag{2.10}$$

Proposition 2.2. *For any integrable function f [7] the integral over the quadratic variation of a* local martingale *is*

$$\int_0^t ds\, a^2(\xi_s)\, f(\xi_s) = \int_0^t ds\, a^2(\xi_s)\, f(\xi_s) \int_{\mathbb{A}} dx\, \delta(x - \xi_s) = \int_{\mathbb{A}} dx\, f(x)\, \ell^{(x)}_{t;\xi} \tag{2.11}$$

where

$$\ell^{(x)}_{t;\xi} = \int_0^t ds\, a^2(\xi_s)\, \delta(x - \xi_s) \tag{2.12}$$

is the local time at x of the local martingale.

Note that

$$\mathrm{E}_{x_o} \ell^{(x)}_{t;\xi} = \int_0^t ds\, a^2(x)\, p_\xi(x, s | x_o) \tag{2.13}$$

so that

$$\frac{d}{dt} \mathrm{E}_{x_o} \ell^{(x)}_{t;\xi} = a^2(x)\, p_\xi(x, t | x_o) \tag{2.14}$$

3 Scale function, speed measure and local time for semimartingales

Let us consider now the diffusion process

$$d\zeta_t = b(\zeta_t)\, dt + c(\zeta_t)\, dw_t \tag{3.1}$$

We look for a change of variables

$$\xi_t = S(\zeta_t) \tag{3.2}$$

where

$$d\xi_t = a(\xi_t)\, dw_t \tag{3.3}$$

In such a case

$$a \circ S(\zeta_t)\, dw_t = \left[d\zeta_t \partial_{\zeta_t} + \frac{1}{2} c^2(\zeta_t) \partial^2_{\zeta_t} \right] S(\zeta_t) \tag{3.4}$$

yields

$$\left[b(z)\, \partial_z + \frac{1}{2} c^2(z)\, \partial_z^2 \right] S(z) = 0 \tag{3.5a}$$

$$a \circ S(z) = c(z) \partial_z S(z) \tag{3.5b}$$

Hence if S is monotonic we have

$$a(x) = (c\, S') \circ S^{-1}(x) \tag{3.6}$$

with $S(x)$ is the *scale measure* of the process

$$S(x) = \int_{x_1}^{x} dy_1\ \exp\{- \int_{x_o}^{y_1} dy_2\, 2\, (c^{-2}\, b)(y_2)\} \tag{3.7}$$

Three observations are in order

- (3.7) is the solution of a second order differential equation: the arbitrariness of the boundary of integration x_o, x_1 stems from the number of boundary conditions that the solution can satisfy.
- A second order differential equation admits two linearly independent solutions: in particular, (3.5a) admits also any constant for solution.
- $S(x)$ is monotonically increasing

$$S'(x) = e^{-\int_{x_o}^{x} dy \frac{2\, b}{c^2}} \geq 0 \tag{3.8}$$

We can construct the local time of the semi-martingale specified by (3.1) following the same steps as in the case of a local martingale in section 2. By (2.9) we have

$$\ell^{(x)}_{t,\xi} = \ell^{(x)}_{t,S(\zeta)} = \int_0^t ds\, (c\, S')^2(\xi_s) \delta(S(\xi_s) - x) \tag{3.9}$$

Thus if we redefine $x \mapsto S(x)$ we obtain

$$\ell^{(x)}_{\tau;\zeta} = \int_0^t ds\, c^2(\xi_s)\, S'(\xi_s)\, \delta(\xi_s - x) \tag{3.10}$$

since

$$\delta(S(y) - S(x)) = \frac{\delta(y - x)}{S'(x)} \tag{3.11}$$

as S' is positive definite.

Definition 3.1. *We define the speed measure as*

$$V_\xi(x) = \frac{1}{c^2(x)\, S'(x)} \tag{3.12}$$

The speed measure enjoys the following properties.

- The average of the local time over the speed measure equals the time elapsed since the time when the initial condition was assigned

$$\int_{\mathbb{R}} dy\, \ell_{\tau;y}^{(x)}\, V_{\xi}(y) = \int_{\mathbb{R}} dy \int_0^t ds\, \delta(y - S^{-1}(x)) = t \tag{3.13}$$

- The speed measure coincides with the stationary measure if this latter exists. Namely

$$\left(\partial_x b - \frac{1}{2}\partial_x c^2\right) p_\star = 0 \tag{3.14}$$

admits the solution

$$p_\star(x) = \frac{e^{\int_{\bar{x}}^x dx_1 \frac{2\,b(x_1)}{c^2(x_1)}}}{c^2(x)} \equiv \frac{1}{c^2(x)\, S'(x)} = V_\xi(x) \tag{3.15}$$

which is an admissible stationary solution if

$$\int_{\mathbb{S}} dx\, p_\star(x) = 1 \tag{3.16}$$

for a suitable choice of $\bar{x}$.

3.1 Examples of scale and speed measure

- Brownian motion with drift:

$$\xi_t = b\,t + a\,w_t \tag{3.17}$$

The scale and speed measure are respectively

$$S'(x) = e^{-\frac{2\,b\,x}{\sigma^2}} \qquad \& \qquad V(x) = \frac{e^{\frac{2\,b\,x}{a^2}}}{a^2} \tag{3.18}$$

- Bessel process

$$d\xi_t = \frac{d-1}{2\,\xi_t} dt + dw_t \tag{3.19}$$

The scale and speed measure are respectively for $d > 1$

$$S'(x) = x^{1-d} \qquad \& \qquad V(x) = x^{d-1} \tag{3.20}$$

3.2 Summary

The local time of a semi-martingale is defined as

$$\ell_{t;\xi}^{(x)} = \int_0^t ds\, \frac{\delta(\xi_s - x)}{V_\xi(\xi_s)} \tag{3.21}$$

The speed measure is the analogous of the inverse flow velocity in the deterministic case. It weights more the points where the process slows down and therefore spends more time. If the process admits a steady state the speed measure coincides with it.

4 Exit time statistics and Feller's classification of boundaries

We restrict the attention to the autonomous case

$$\mathfrak{L}_x = b(x)\partial_x + \frac{1}{2}a^2(x)\partial_x^2 \tag{4.1}$$

Let $I = (x_m, x_M)$ an open interval. We define the first *hitting* time of x_m as

$$\tau_{x|x^m} = \inf_{t\in\mathbb{R}_+} \{t : \xi_t = x_m \quad \& \quad \xi_0 = x\} \tag{4.2}$$

and of x_M

$$\tau_{x|x_M} = \inf_{t\in\mathbb{R}_+} \{t : \xi_t = x_M \quad \& \quad \xi_0 = x\} \tag{4.3}$$

The exit time from the interval is then

$$\tau_x = \tau_{x|x_m} \wedge \tau_{x|x_M} \tag{4.4}$$

4.1 Scale measure and hitting probability

The probability of exiting I from x_m admits an explicit expression in terms of the scale measure

$$\mathrm{P}(\xi_{\tau_x} = x_m) = \mathrm{P}(\tau_{x|x_m} < \tau_{x|x_M}) = \frac{S(x_M) - S(x)}{S(x_M) - S(x_m)} \tag{4.5}$$

provided the scale measure satisfies $-\infty < S(x_m) \leq S(x_M) < \infty$. From (4.5) we can draw the following conclusions.

- If the scale measure blows up at x_m then the diffusion hits first x_M with probability one.

$$\begin{cases} \lim_{x\searrow x_m} S(x) = -\infty \\ S(x_M) < \infty \end{cases} \qquad \Rightarrow \qquad \mathrm{P}(\xi_{\tau_x} = x_m) = \mathrm{P}(\tau_{x|x_m} < \tau_{x|x_M}) = 0 \tag{4.6}$$

- If the scale measure is finite at x_m and diverges at x_M the diffusion hits first the lower boundary with probability one

$$\begin{cases} S(x_m) > -\infty \\ \lim_{x\nearrow\infty} S(x) = +\infty \end{cases} \qquad \Rightarrow \qquad \mathrm{P}(\xi_{\tau_x} = x_m) = \mathrm{P}(\tau_{x|x_m} < \tau_{x|x_M}) = 1 \tag{4.7}$$

In order to understand these results we observe that by its very definition (3.2) the scale measure maps a semi-martingale into a Wiener process parametrized by a random clock. Thus the divergence of the scale measure corresponds to the Wiener process diffusing to plus or minus infinity. The probability of such events is infinitesimal since it is specified by the tails of a Gaussian distribution. Hence the results above must follow.

4.2 Expected exit time

Over an infinite time horizon the expected value of the exit time from I satisfies

$$\mathfrak{L}_x \, \mathrm{E}\tau_x = -1 \tag{4.8a}$$

$$\mathrm{E}\,\tau_{x_m} \,=\, \mathrm{E}\,\tau_{x_M} = 0 \tag{4.8b}$$

We can couch the solution into the form

$$\mathrm{E}\,\tau_x = \mathrm{P}(\xi_\tau = x_m)\,\mathrm{E}\,\tau_{x|x_m} + \mathrm{P}(\xi_\tau = x_M)\,\mathrm{E}\,\tau_{x|x_M} \tag{4.9}$$

with

$$\mathrm{E}\tau_{x|x_m} \equiv \mathrm{E}\,\{\tau_x|\xi_\tau\} = 2\int_{x_m}^{x} dy\,V(y)\,[S(y) - S(x_m)] \tag{4.10a}$$

$$\mathrm{E}\tau_{x|x_m} \equiv \mathrm{E}\,\{\tau_x|\xi_\tau = x_M\} = 2\int_{x}^{x_M} dy\,V(y)\,[S(x_M) - S(y)] \tag{4.10b}$$

Note that whenever they are finite over $[x_m, x_M]$ the functions $\mathrm{E}\tau_{x|x_m}$, $\mathrm{E}\tau_{x|x_M}$ enjoy the following properties.

- They are positive definite

$$\mathrm{E}\tau_{x|x_m}\,, \mathrm{E}\tau_{x|x_M} \,\geq\, 0 \tag{4.11}$$

- They vanish on the conditioning boundaries

$$\mathrm{E}\,\tau_{x_m|x_m} = \mathrm{E}\,\tau_{x_M|x_M} \,=\, 0 \tag{4.12}$$

- They admit the re-writing

$$\mathrm{E}\tau_{x|x_m} = 2\int_{x_m}^{x} dy\,V(y)\int_{x_m}^{y} dy_1\,S'(y_1) = 2\int_{x_m}^{x} dy\,S'(y)\int_{y}^{x} dy_1\,V(y_1) \tag{4.13a}$$

$$\mathrm{E}\tau_{x|x_M} = 2\int_{x}^{x_M} dy\,V(y)\int_{y}^{x_M} dy_1\,S'(y_1) = 2\int_{x}^{x_M} dy\,S'(y)\int_{x}^{y} dy_1\,V(y_1) \tag{4.13b}$$

The expressions to the right are natural to resort to in order to study the boundary behavior as a limit:

$$\lim_{\varepsilon\searrow 0}\mathrm{E}\tau_{x|x_m+\varepsilon} = 2\int_{x_m+\varepsilon}^{x} dy\,S'(y)\int_{y}^{x} dy_1\,V(y_1) \tag{4.14a}$$

$$\lim_{\varepsilon\searrow 0}\mathrm{E}\tau_{x|x_M-\varepsilon} = 2\int_{x}^{x_M-\varepsilon} dy\,S'(y)\int_{x}^{y} dy_1\,V(y_1) \tag{4.14b}$$

4.3 Interpretation of the exit time integrand

The stationary state equation

$$(-\partial_x b + \frac{1}{2}\partial_x^2 a^2)\mathrm{p} = 0 \tag{4.15}$$

equation admits beside the aforementioned solution $\mathrm{p}_1(x) = V(x)$ the solution

$$\mathrm{p}_2(x) = c\,V(x)\int_{x_o}^{x} dy\,S'(y) \equiv c\,V(x)\int_{x_o}^{x} dy\,\frac{1}{V(y)\,a^2(y)} \tag{4.16}$$

corresponding to the *constant flux* condition

$$\left(-b + \frac{1}{2}\partial_x a^2\right)\,\mathrm{p} = c \tag{4.17}$$

with c and x_o two constants to be fixed by imposing the boundary conditions.

5 Feller classification of boundaries

Let us fix the attention on x_m. A nice presentation of Feller's classification can be found in [3] pag. 293 (or [2] pag 231). It is convenient to define

$$b(x) = -\, a^2(x)\, (\partial_x U)(x) \tag{5.1}$$

This means that if g specifies the inverse of a metric then b is the *gradient vector field* (push-forward) of the potential U. The definition (5.1) yields for the scale function

$$S(x; x_o, x_1) = \int_{x_1}^{x} dy\, e^{2\,[U(y)-U(x_o)]} \tag{5.2}$$

and

$$V(x; x_o) = \frac{e^{-2\,[U(x)-U(x_o)]}}{a^2(x)} \tag{5.3}$$

5.1 Diffusion coefficient degenerate at one boundary

Suppose that diffusion coefficient vanishes at x_m while satisfying there the Lipschitz condition required by the xistence and uniqueness theorem

$$a(x_m) = 0 \qquad \& \qquad (\partial_x a^2)(x_m) = 0$$

- Exit boundary:

  $$b(x_m)\; <\; 0 \tag{5.4}$$

  if the diffusion reaches x_m it will leave I with unit probability.

- Entrance boundary:

  $$b(x_m)\; >\; 0 \tag{5.5}$$

  if the diffusion reaches x_m the drift will steer it back into I.

- Natural boundary:

  $$b(x_m)\; =\; 0 \tag{5.6}$$

  x_m is a fixed point for the diffusion process.

For example let us consider

$$d\xi_t = (\mu - \sigma^2\, \xi_t)\, dt + \sigma\, \xi_t(1 - \xi_t)\, dw_t \tag{5.7}$$

for $\xi_t \in (0, 1)$.

- the diffusion coefficient vanishes for

  $$\xi_t = 0 \qquad \& \qquad \xi_t = 1 \tag{5.8}$$

- The drift is positive in zero if $\mu\; >\; 0$

- The drift is negative in one if $\mu < \sigma^2$

The stationary distribution is

$$p_\xi(x) = \frac{C}{x^{4-\frac{4\mu}{\sigma^2}}(1-x)^{\frac{4\mu}{\sigma^2}}} \exp\left\{-2\frac{\mu + x(\sigma^2 - 2\mu)}{\sigma^2 x (1-x)}\right\} \tag{5.9}$$

and is always normalisable for

$$0 < \mu < \sigma^2 \tag{5.10}$$

as the argument of the exponential is negative definite.

5.2 General Feller classification

In general, the boundary x_m may be of four different types. To classify them let us fix an arbitrary $\bar{x}$ such that $x_m < \bar{x} < x_M$.

5.2.1 Regular boundary

$$\int_{x_m}^{\bar{x}} dy\, V(y) < \infty \qquad \& \qquad S(x_m) > -\infty \tag{5.11}$$

There is a finite probability to hit x_m in finite time and the speed measure is integrable. Note that the conditions above also imply (the scale measure grows monotonically)

$$\mathrm{E}\,\tau_{x|x_m} < \infty \tag{5.12}$$

It is possible to impose reflecting boundary conditions at the boundary.

5.2.2 Exit (or capturing boundary) boundary

$$\int_{x_m}^{\bar{x}} dy\, V(y) = \infty \qquad \& \qquad \mathrm{E}\,\tau_{x|x_m} < \infty \tag{5.13}$$

the boundary can be reached, on average, in a finite time. The speed measure is not integrable entailing that the process does not admit a normalizable steady state because the diffusion accumulates at x_m (i.e. it “exits” the open domain (x_m, x_M)). For example, Let us suppose

$$x_m = 0 \tag{5.14}$$

and

$$\begin{cases} b(x) = b_o < 0 \\ a^2(x) = a_o^2\, x^2 \end{cases} \quad \Rightarrow \quad U(x) = \frac{b_o}{a_o^2\, x}$$

We have then

$$S'(x) \propto e^{-\frac{2\,|b_o|}{a_o^2\, x}} \qquad \& \qquad V(x) \propto \frac{e^{\frac{2\,|b_o|}{a_o^2\, x}}}{a_o^2\, x^2}$$

Hence the speed measure is not integrable at the origin for any $b_o \leq 0$. In order to estimate the conditional expectation of the exit time we recall the *first mean value theorem for definite integrals*

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

This means that there exists some $\bar{y}(y) \in [0, y]$ such that

$$\mathrm{E}\tau_{x|0} = \int_0^x \mathrm{d}y\, \frac{y\, S'(\bar{y})}{a^2(y)\, S'(y)} = \int_0^x \mathrm{d}y\, \frac{e^{\frac{2\,|b_o|\,(\bar{y}-y)}{a_o^2\, y\, \bar{y}}}}{a_o^2\, y} \,<\, \infty$$

In this case one can impose absorbing or constant flux boundary conditions (i.e. re-inject from the other boundary the probability loss).

5.2.3 Entrance boundary

$$\int_{x_m}^{\bar{x}} dy\, V(y) \,<\, \infty \qquad \& \qquad \mathrm{E}\,\tau_{x|x_m} \,=\, \infty \tag{5.15}$$

Contrasting this condition with (4.10a) we infer that

$$\lim_{x \downarrow x_m} S(x) = -\infty \qquad \Rightarrow \qquad \mathrm{E}\,\tau_{x|x_m} \,=\, \infty \tag{5.16}$$

For example let at $x_m = 0$ the drift and diffusion satisfy

$$\begin{cases} b(x) = b_o \,>\, 0 \\ a^2(x) = a_o^2\, x^2 \end{cases} \qquad \Rightarrow \qquad U(x) = \frac{b_o}{a_o^2\, x}$$

It follows that

$$S'(x) \propto e^{\frac{2\,b_o}{a_o^2\, x}} \qquad \& \qquad V(x) \propto \frac{e^{-\frac{2\,b_o}{a_o^2\, x}}}{a_o^2\, x^2}$$

Hence the speed measure is integrable at the origin but

$$\mathrm{E}\tau_{x|0} = \int_0^x \mathrm{d}y\, \frac{y\, S'(\bar{y})}{a^2(y)\, S'(y)} = \int_0^x \mathrm{d}y\, \frac{e^{\frac{-2\,b_o\,(\bar{y}-y)}{a_o^2\, y\, \bar{y}}}}{a_o^2\, y} \,=\infty$$

5.2.4 Natural boundary

We have

$$\int_{x_m}^{\bar{x}} dy\, V(y) \,=\, \infty \qquad \& \qquad \mathrm{E}\,\tau_{x|x_m} \,=\, \infty \tag{5.17}$$

For example, for $x_m = 0$

$$\begin{cases} b(x) = b_o\, x^2 \\ a^2(x) = a_o^2\, x^2 \\ b_o \,>\, 0 \end{cases} \qquad \Rightarrow \qquad U(x) = -\frac{b_o x}{a_o^2} \tag{5.18}$$

In such a case, we have

$$S'(x) \propto e^{-\frac{2\,b_o\,x}{a_o^2}} \qquad \& \qquad V(x) \propto \frac{e^{\frac{2\,b_o\,x}{a_o^2}}}{a_o^2\,x^2} \tag{5.19}$$

whence

$$\mathrm{E}\,\tau_{x|\varepsilon} \sim \int_{\varepsilon}^{\bar{x}} dy\, \frac{e^{\frac{2\,b_o\,x}{a_o^2}} - 1}{a_o^2\,x^2} \overset{\varepsilon\downarrow 0}{\to} \infty \tag{5.20}$$

5.3 Behavior at a reachable boundary

We follow here the terminology of [5] pag. 167

- Absorbing boundary:

$$\xi_t = r_i \qquad \forall\, t\, >\, \tau_x \tag{5.21}$$

- Instantaneous reflection: for ε arbitrarily small

$$\xi_{\tau_x+\varepsilon}\, \in\, (x_m, x_M) \tag{5.22}$$

  Furthermore the event

$$A = \{t\colon \xi_t = x_m\} \cup \{t\colon\; \xi_t = x_M\} \tag{5.23}$$

  has *zero* Lebesgue measure.

- Slow reflection (sticky boundary): for ε arbitrarily small

$$\xi_{\tau_x+\varepsilon}\, \in\, (x_m, x_M) \tag{5.24}$$

  Furthermore the event (5.23) has *finite* Lebesgue measure.

- Jumps at the boundary: re-injection with some probability once the boundary is reached.

In the ensuing section we will give some quantitative criteria to discriminate between the different boundary behaviors.

6 Example: Cox–Ingersoll–Ross process

Feller introduced the process which is now know as CIR (Cox–Ingersoll-Ross process) in [4] as a model problem for the study of boundary behavior of stochastic differential equations. The process was afterwards re-derived in [1] where it came about as a model of interest rates dynamics.

The process is described by the Ito stochastic differential equation

$$d\xi_t = (c + b\,\xi_t)\,dt + \sqrt{2\,a\,\xi_t}\,dw_t \tag{6.1}$$

The equivalent Stratonovich stochastic differential equation is

$$d\xi_t = (c - a + b\,\xi_t)\,dt + \sqrt{2\,a\,\xi_t}\,dw_t \tag{6.2}$$

6.1 Elementary considerations

- First moment:

$$\mathrm{E}\xi_t = e^{b\,t}\mathrm{E}\xi_0 + \frac{c}{b}\left(e^{b\,t} - 1\right) \tag{6.3}$$

Thus for ξ_0 close to the origin and $b\,t \ll 1$ we have

$$\mathrm{E}\xi_t \sim \mathrm{E}\xi_0 + c\,t \tag{6.4}$$

which hints that the origin is attractive for $c\,<\,0$. Stratonovich representation states instead that path close to differentiable ones behave as

$$\mathrm{E}\xi_t \sim \mathrm{E}\xi_0 + (c - a)\,t \tag{6.5}$$

provided

$$b\,t\,\ll\,1 \qquad \& \qquad \sqrt{2\,a\,b}\,t\,\ll\,1 \tag{6.6}$$

- Second moment:

$$d\mathrm{E}\xi_t^2 = 2\,[(c + a)\,\mathrm{E}\xi_t + b\,\mathrm{E}\xi_t^2]\,dt \tag{6.7}$$

whence

$$\mathrm{V}\xi_t = \mathrm{E}\xi_t^2 - (\mathrm{E}\xi_t)^2 \tag{6.8}$$

satisfies

$$d\mathrm{V}\xi_t = 2\,(a\,\mathrm{E}\xi_t + b\,\mathrm{V}\xi_t)\,dt \tag{6.9}$$

We obtain

$$\mathrm{V}\xi_t = \mathrm{V}\xi_o\,e^{2\,b\,t} + \frac{2\,a\,\mathrm{E}\xi_o}{b}(e^{b\,t} - 1)\,e^{b\,t} + \frac{2\,a\,c}{b^2}(e^{b\,t} - 1)^2 \tag{6.10}$$

For small values of t

$$\mathrm{V}\xi_t = \mathrm{V}\xi_o + 2\,(a\,\mathrm{E}\xi_o + b\,\mathrm{V}\xi_o)\,t + O(t^2) \tag{6.11}$$

6.2 Boundary behavior

We have bay definition $a\,>\,0$

$$U(x) = -\int^x dy\,\frac{c + b\,y}{2\,a\,y} = \frac{c}{2\,a}\ln\frac{1}{x} - \frac{b\,x}{2\,a} \tag{6.12}$$

whence

$$S'(x) = \left(\frac{1}{x}\right)^{\frac{c}{a}} e^{-\frac{b\,x}{a}} \qquad \& \qquad V(x) \propto x^{\frac{c}{a}-1} e^{\frac{b\,x}{a}} \tag{6.13}$$

and

$$\mathrm{E}\tau_{x|\varepsilon} \sim \int_\varepsilon^{\bar{x}} \frac{dy}{y}\,\frac{e^{-\frac{by}{a}}}{y^{\frac{c}{a}-1}} \int_y^{\bar{x}} dy_1\,y_1^{\frac{c}{a}-1} e^{\frac{b\,y_1}{a}} \tag{6.14}$$

- $c \le 0$: Feller proved that the initial condition fully specify a solution of the Fokker-Planck equation without the need of a boundary condition at $x = 0$. Such solution is positivity but not norm preserving. By (6.13) we see that the speed measure is not integrable at the origin. The conditional expected exit time is, however finite. The origin is a *capturing boundary*.
- $0 < c \le a$: the origin is a regular boundary.
  - it is possible to impose a reflecting boundary condition at the origin. The solution of the Fokker-Planck equation is in such a case positivity and norm preserving.
  - there exist "generalised" absorbing boundary conditions. Such solutions are positivity but not norm preserving.
- $c > a$: the speed measure is integrable at the origin but the speed measure thereby diverges. The origin is not reachable. Hence there exists a norm and positivity preserving solution for which both flux and probability density vanish at the origin.

Feller shows that *reflecting boundary* at zero for $0 < c \le a$ yield

$$\lim_{x \searrow 0} p(x,t) = \begin{cases} \infty & 0 < c < a \\ \text{finite positive} & c = a \\ 0 & c > a \end{cases} \qquad \& \qquad j(0,t) = 0 \tag{6.15}$$

This means that for $c > a$ an outward flux condition in the origin would give rise to a non-positive definite solution of the Fokker-Planck equation.

7 Foguel alternative and ergodic results

Suppose that the solution of the $\mathbb{D}$ valued Itô time-autonomous stochastic differential equation

$$d\boldsymbol{\xi}_t = \boldsymbol{b}(\boldsymbol{\xi}_t)\mathrm{d}t + \mathsf{A}(\boldsymbol{\xi}_t) \cdot d\boldsymbol{w}_t \tag{7.1}$$

is sufficiently regular to admit a unique stationary measure $\rho_\star$ over $\mathbb{D}$. Foguel alternative is a general result about the time-asymtpotic properties of solutions of the time-autonomous backward Kolmogorov equation. It states that

Proposition 7.1. *For any integrable test function with respect to the measure specified by (7.1), either of the two statements below holds (see § 3.2.4 of [6] for details):*

1. *the time average of the expectation value vanishes*

$$\lim_{T \uparrow \infty} \frac{1}{T} \mathrm{E}_{\boldsymbol{x},0} \int_0^T \mathrm{d}t\, f(\boldsymbol{\xi}_t) = 0$$

2. *the time average of the expectation value tends to the average with respect to the stationary measure $\rho_\star$*

$$\lim_{T \uparrow \infty} \frac{1}{T} \mathrm{E}_{\boldsymbol{x},0} \int_0^T \mathrm{d}t\, f(\boldsymbol{\xi}_t) = \int_{\mathbb{D}} \mathrm{d}\boldsymbol{y}\, \rho_\star(\boldsymbol{y})\, f(\boldsymbol{y})$$

We avail us of Foguel alternative to relate the time-asymtpotic properties of solutions of the time-autonomous backward Kolmogorov equation to ergodic theory. Namely we may construct a function ϕ such that

$$\mathfrak{L}_{\boldsymbol{x}}\,\phi = f \tag{7.2a}$$

$$\phi|_{\boldsymbol{x}\in\partial\mathbb{D}} = \text{b.c. depending upon the stochastic process} \tag{7.2b}$$

then by Dynkin formula

$$\int_0^T dt\, f(\boldsymbol{\xi}_t) = \int_0^T dt\,(\partial_t + \mathfrak{L}_{\boldsymbol{\xi}_t})\phi(\boldsymbol{\xi}_t, t) = \phi(\boldsymbol{\xi}_T, T) - \phi(\boldsymbol{\xi}_0, 0) - \int_0^T [\mathsf{A}(\boldsymbol{\xi}_t)\cdot d\boldsymbol{w}_t]\cdot\partial_{\boldsymbol{\xi}_t}\phi(\boldsymbol{\xi}_t) \tag{7.3}$$

we have the general relation

$$\int_0^T dt\,\mathrm{E}_{\boldsymbol{x}} f(\boldsymbol{\xi}_t) = \mathrm{E}_{\boldsymbol{x}}\phi(\boldsymbol{\xi}_T, T; T) - \phi(\boldsymbol{x}, 0; T) \tag{7.4}$$

and consequently

$$\lim_{T\nearrow\infty}\frac{\mathrm{E}_{\boldsymbol{x}}\phi(\boldsymbol{\xi}_T, T; T) - \phi(\boldsymbol{x}, 0; T)}{T} = \lim_{T\nearrow\infty}\frac{1}{T}\int_0^T dt\,\mathrm{E} f(\boldsymbol{\xi}_t) \tag{7.5}$$

where the average is with respect to the invariant measure of $\boldsymbol{\xi}_t$. This means that the average evolves as

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}})\,\phi = f \tag{7.6a}$$

$$\phi(\boldsymbol{x}, T) = \phi_o(\boldsymbol{x}) \tag{7.6b}$$

If ϕ_o is bounded we obtain

$$\lim_{T\nearrow\infty}\frac{\phi(\boldsymbol{x}, 0; T)}{T} = -\lim_{T\nearrow\infty}\frac{1}{T}\int_0^T dt\,\mathrm{E}_{\boldsymbol{x}} f(\boldsymbol{\xi}_t) \tag{7.7}$$

7.1 Slow reflection in $1d$: diffusion coefficient vanishing at one boundary

Suppose first

$$a(0) = 0 \tag{7.8}$$

and consider a diffusion in an interval $I \subseteq \mathbb{R}_+$ with left boundary in the origin. We also suppose that the speed measure is integrable.

- Exit boundary:

$$b(0) < 0 \tag{7.9}$$

- Entrance boundary:

$$b(0) > 0 \tag{7.10}$$

- Natural boundary:

$$b(0) = 0 \tag{7.11}$$

We have the boundary condition

$$\lim_{x\searrow 0}[(\partial_t + b\,\partial_x)\phi](x,t) = f(0) \tag{7.12}$$

By definition the speed measure satisfies

$$\mathfrak{L}_x^{\dagger} V = 0 \tag{7.13}$$

we have

$$\int_{\mathbb{R}_+} dx\, V(x)\,[(\partial_t + \mathfrak{L}_x)\phi](x,t) = \int_{\mathbb{R}_+} dx\,(V\,f)(x) \tag{7.14}$$

Integrating by parts we get into

$$-[V\,(b\,+a\partial_x)\phi](0) + [(\phi\partial_x)a\,V](0) + \int_{\mathbb{R}_+} dx\,(V\partial_t\phi)(x,t) = \int_{\mathbb{R}_+} dx\,(V\,f)(x) \tag{7.15}$$

Let us suppose

$$\lambda := -\lim_{T\nearrow+\infty} \partial_t\phi(x,t;T) = \lim_{T\nearrow+\infty} \frac{\phi(x,t;T)}{T} \equiv -\lim_{T\nearrow\infty} \frac{1}{T}\int_0^T dt\,\mathrm{E}_{\boldsymbol{x}} f(\boldsymbol{\xi}_t) \tag{7.16a}$$

$$\int_{\mathbb{R}_+} V(x) = K\,<\,\infty \tag{7.16b}$$

with λ, K constants. The probability flux at zero is then

$$-[\phi\,(b - \partial_x)a\,V](0) := \Phi \tag{7.17}$$

The equation satisfied by λ is

$$\Phi - \left[\left(\frac{V\,a}{b}\right)(0) + K\right]\lambda - \left(\frac{V\,a\,f}{b}\right)(0) = \int_{\mathbb{R}_+} dx\,(V\,f)(x) \tag{7.18}$$

which yields

$$\lambda = -\frac{f(0) + b(0)\,\left[\int_{\mathbb{R}_+} dx\,(V\,f)(x) - \Phi\right]}{1 + b(0)\int_{\mathbb{R}_+} dx\,V(x)} \tag{7.19}$$

and therefore

$$\lim_{T\nearrow\infty} \frac{1}{T}\int_0^T dt\,\mathrm{E}_{\boldsymbol{x}} f(\boldsymbol{\xi}_t) = \frac{f(0) + b(0)\,\left[\int_{\mathbb{R}_+} dx\,(V\,f)(x) - \Phi\right]}{1 + b(0)\int_{\mathbb{R}_+} dx\,V(x)} \tag{7.20}$$

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